LVIII. *A treatise on the precession of the equinoxes, and in general on the motion of the nodes, and the alteration of the inclination of the orbit of a planet to the ecliptic. Inscribed to the gentlemen of the* Royal Society, *by M.* De St. Jaques Silvabelle.

Translated from the French *M. S. by* J. Bevis, *M. D.*

INTRODUCTION.

Read March 12, 1752. IF the earth were perfectly spherical, the action of the sun on all the parts which compose it, would not produce any effect to make it turn round its centre; because the moment, which would be produced on one side, would be always counterbalanced by an equal moment on the opposite side of the centre.

It would be the same, if the earth were a spheroid flatted at the poles, and the sun was always in the equator, or in the ninetieth degree of declination: But in every other degree of declination its action on the excess of matter about the equator has a tendency to make the equator approach towards the sun's place, or to diminish the angle of the sun's declination, by making the earth's axis to turn round its centre in the plane of the circle of the sun's declination.

The earth has then, at every instant, two motions of rotation; one about the axe of the equator, called also the earth's axe; and this is the diurnal motion,

C c c   which

which is uniform; the other motion of rotation is performed about the axe of the circle of the fun's declination, which is a diameter of the equator; and this motion is produced by the action of the fun on the redundant matter about the equator, and is continually accelerated, from the continual application of the folar action producing it.

The point E, *fig.* 1. *n°* 2. which is the interfection of the circumference of the equator and the circumference of the circle of declination, has two motions, whofe directions are perpendicular to each other. Let $E e$ be the fpace, which it runs through in an inftant $d t$, in the circumference of the equator, by the uniform diurnal motion, and let $E \varepsilon$ be the fpace it runs through in the fame inftant, in the circumference of the circle of declination, by an accelerated motion, as has been explained.

The point E, in vertue of thefe two motions $E e$ and $E \varepsilon$, will not circulate either in the circumference $E e \mathcal{Q} e' \mathcal{Q} E$ of the equator, or in the circumference $E \varepsilon P E' P' E$ of the circle of declination, but forming the rectangular parallelogram $E e e' \varepsilon$, the diagonal $E e'$ will be the elementary arc of the circumference $E e q E'$, in which the point E will circulate, and the angle $e E e'$ will be equal to the angle $\mathcal{Q} C q$, and equal to the angle $P' C p'$ which the pole P' runs through in an inftant in the circumference $P' p' \mathcal{Q} q P \mathcal{Q}' P'$ whofe plane is perpendicular to the plane of the circle of declination; and when the lines $E e$, $E \varepsilon$ are known at every inftant, $P' p'$ will alfo be known, fince the angle $e E e' =$ to the angle $\mathcal{Q} C q =$ to the angle $P' C p'$.
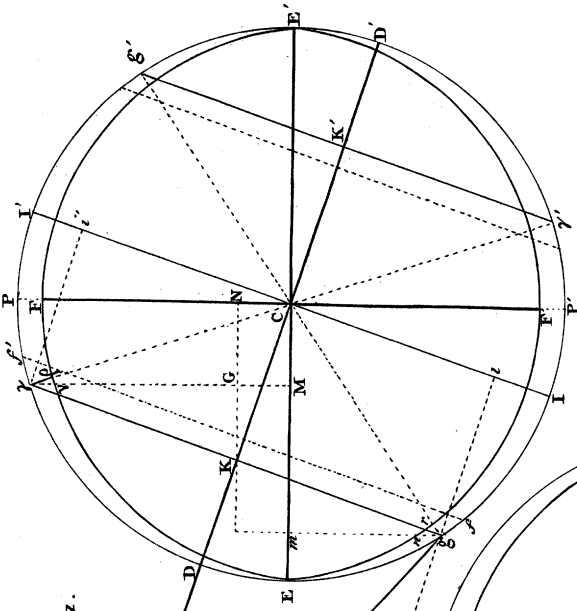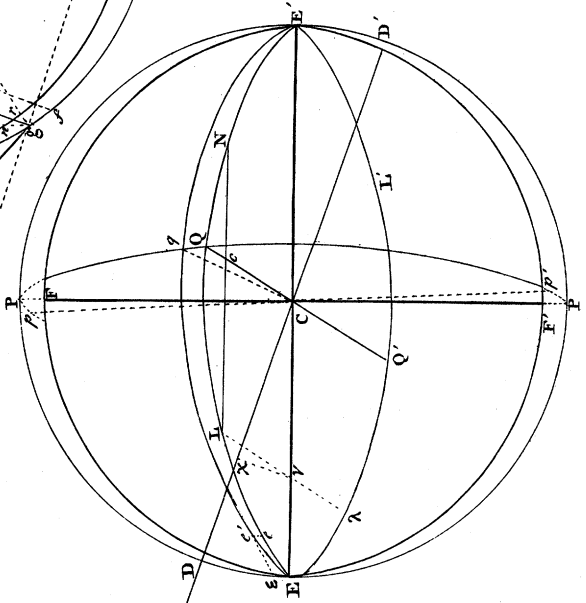
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Fig. 1. N°. 1.
p. 392.

Fig. 1. N°. 2.
p. 386. &c.

The inftantaneous motion of the pole, which is $P' p'$, or $P p$, *fig*. 2. n° 1. & n^q 2 *. may be refolved into two, $P R$ and $P M$, perpendicular to each other, and both to the earth's axe. The former caufes the pole P to move parallel to the ecliptic ♈ ♋ ♎ ♑, and alters the place of the folftice ♋, and confequently alfo that of the equinoctial points ♈ and ♎; the latter, which is according to $P M$, alters the inclination of the earth's axe to the ecliptic.

To have the motion of the pole parallel to the ecliptic, or, which is the fame, the motion of the node ♈, or the preceffion, in the fame time that the fun paffes from the equinox ♈ to the folftice ♋, take the integral of the lines $P R$, fuppofing $P R$ generally to exprefs the inftantaneous preceffion for any declination of the fun S.

And to have the alteration of the inclination in the fame time that the fun is paffing from ♈ to ♋, take the fum, or the integral of the lines $P M$, fuppofing $P M$ generally to exprefs the inftantaneous alteration of the inclination of the earth's axe to the ecliptic for any given declination of the fun.

The fum of the lines $P R$ is always the fame, and has the fame fign, or the fame direction, during every quarter of the fun's revolution, whether he moves from ♈ to ♋, or from ♋ to ♎, or from ♎ to ♑, or from ♑ to ♈; fo that the preceffion anfwering to any one quarter of the fun's revolution about the earth, or to three months, being known, that multiplied by 4 will be the annual preceffion; by 8 will give it for two years; by 16 for 4 years, &c.

Likewife

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* *Fig*. 2. is explained at the beginning of *Prob*. 5.

Likewife the fum of the lines *P M* is ever the fame for every quarter of the folar revolution; but it has alternatively a contrary fign; that is, a contrary direction. During the quarter from ♈ to ♋, the alteration of the inclination of the earth's axe to the ecliptic is pofitive, and the angle of the inclination increafes; but during the fucceeding quarter, or from ♋ to ♎, the alteration of the inclination is negative, and the angle of the inclination diminifhes: And as the diminution from ♋ to ♎ is equal to the augmentation from ♈ to ♋, it follows, that at the end of the femirevolution the inclination of the earth's axe to the plane of the ecliptic will become again the fame, having undergone an ofcillation, which is completed in a femirevolution. It is the fame, when the fun paffes from ♎ to ♈. The angle of the inclination increafes from ♎ to ♑, and decreafes from ♑ to ♈, where it becomes again the fame it was at ♎.

And hence the inclination of the earth's axe to the ecliptic may be confidered as conftant, tho' fubject to this ofcillation, and indeed to feveral others, which will be prefently explained, they being all regular, and performed in regular periods.

The earth's inclination to the ecliptic being conftant, and the motion of the pole which produces the preceffion, being always parallel to the plane of the ecliptic, the earth's pole moves in a parallel to the ecliptic, about 23 degrees and a half diftant from the pole of the ecliptic, and the terreftrial axe defcribes a conic furface.

To this motion of the terreftrial axe or pole is to be afcribed the apparent motion of the ftars about the pole of the ecliptic.

But

But hitherto we have not confidered, that to the preceffion, thus caufed by the fun, we are to add that likewife produced by the moon; and it remains, that we examine into the motion of the earth's pole, caufed by the action of the moon on the redundant matter about the earth's equator.

All, that has been faid concerning the fun, is alike applicable to the moon, which we may put in the place of the fun; the moon's orbit in the place of the ecliptic; and the time of the moon's revolution round the earth in the place of the revolution of the fun round the earth: And we fhall find the motion of the earth's pole parallel to the lunar orbit, which is always the fame at every quarter of the time of the revolution of the moon round the earth, and the ofcillation of the earth's axe to the plane of the lunar orbit, which is completed in each femirevolution of the moon round the earth.

But whereas the plane of the lunar orbit, which is always inclined to the plane of the ecliptic in an angle of about 5 degrees, never continues in a conftant pofition, like the plane of the ecliptic, fo that its pole defcribes a fmall circle parallel to the ecliptic, at the diftance of about 5 degrees from the pole thereof; it follows, that the preceffion, with refpect to the lunar orbit, is not the fame as with refpect to the ecliptic; and that the motion of the pole parallel to the lunar orbit fhould be referred to the plane of the ecliptic: Which is done by refolving the motion of the pole, parallel to the plane of the lunar orbit, into two motions, the one parallel to the plane of the ecliptic, and the other perpendicular thereto, and in the plane of the folfticial colure.

The

The former of thefe two motions gives the preceffion with refpect to the ecliptic, and has its direction always the fame way.

The latter motion has two oppofite directions, in the two femirevolutions of the pole of the lunar orbit round the pole of the ecliptic, and caufes an ofcillation of the terreftrial axe on the plane of the ecliptic, which is compleated in a revolution of the pole of the lunar orbit round the pole of the ecliptic.

From all that has been faid, it follows, that there are five diftinct motions of the pole of the earth; namely, two of preceffion, which are parallel to the plane of the ecliptic, and three of ofcillation on the plane of the ecliptic.

The two of preceffion are caufed, the one by the fun, the other by the moon. That, which is caufed by the fun, is conftantly the fame at every quarter of the time of the revolution of the fun round the earth, that is, every three months: That which is caufed by the moon, is conftantly the fame at every quarter of the time of the revolution of the moon round the earth; that is, about every feven days.

Of the three motions of ofcillation, one is caufed by the fun, and is completed in the time of the femirevolution of the fun round the earth, taken from one equinox to the following one; that is, in fix months.

The other is caufed by the moon; and each ofcillation is completed in the fpace of a femirevolution of the moon round the earth; that is, in about 14 days.

The third is caufed likewife by the moon, and arifes from the plane of her orbit being different from the plane of the ecliptic, and from the pole of the lunar orbit making its revolution about the pole of the
ecliptic

ecliptic in about 18 years and two thirds. And this ofcillation is compleated in the time of the revolution of the pole of the lunar orbit about the pole of the ecliptic; that is, in about 18 years and two thirds.

It will appear in the memoir, that there is a relation purely geometrical between the quantity of the nutation, during the time of the femirevolution of the pole of the lunar orbit, and the quantity of the preceffion, caufed likewife by the moon in the fame time. This relation is quite independent of the force of the moon, of the quantity of the earth's flatnefs, of the quantity of the terreftrial matter, and, in a word, of every thing of a phyfical nature that can enter into the problem.

We are content to examine the motions of the pole of the earth produced by the fun and the moon. The fame method, and the fame *formulæ*, will give likewife the motions of the terreftrial pole arifing from any other planet, as Saturn, Jupiter, &c. but thefe motions are too minute to merit attention.

Whatever has been faid of the action of the fun on the redundant matter about the earth's equator, is alfo applicable to his action on a fimple ring placed at the equator, without adhering to the terreftrial globe; and the motion of the pole of fuch ring may be determined by the fame method, and confequently the motion of its nodes on the plane of the ecliptic, and the alteration of the inclination of its axe to the fame plane. And fince thefe motions are the fame, whether the ring be fuppofed entire, or a fmall portion of it only be confidered, or a mere point thereof, the motions of the nodes, and the alteration of the inclination of a moon, or a fatellite of a planet, may
thereby

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thereby be known. And the *formulæ* differ in nothing from thofe of the motion of the nodes of the earth's equator, and of the alteration of the obliquity of the earth's axe to the plane of the ecliptic, but in this; that the action of the fun on the ring to make it turn, is exerted entirely thereon; whereas in the problem of the preceffion this force muft neceffarily be diftributed throughout the whole mafs of the earth, on account of the adherence of the ring to the globe of the earth.

DIVISION of the WORK.

This memoir is divided into four fections.

The 1ft fection treats of the motion of the pole of the terreftrial equator caufed by the fun.

The 2d fection treats of the motion of the pole of the terreftrial equator caufed by the moon.

The 3d fection treats of the motion of the pole of a ring, or of the orbit of a moon, caufed by the fun.

The 4th fection contains the application of the *formulæ* found in the other fections.

SECT. I.

Of the motion of the pole of the terreftrial equator caufed by the action of the fun.

PROBLEM I.

Article 1. *To find the moment, which refults from the attraction of each particle of the earth towards the fun, in the inverfe ratio of the fquare of the diftance, to make the earth's axis turn upon its centre C.*

Let *E F E' F E, fig.* 1. *n°* 1. be the fection of the earth by the plane of the circle of the fun's declination ;

tion; $P E P' E' P$ the circumference whofe diameter is the earth's equator, $E E'$ or $P P'$. Let $E E'$ be perpendicular to $P P'$, and in the equator. Let $I I'$ be perpendicular to $C S$, and affume this diameter $I I'$ for the leaver to which all the moments are to be referred.

The motion of any point, as g, towards the fun, is $\dfrac{S}{\overline{S g}^2}$. By refolving this motion into two, one according to $g C$, the other parallel to $C S$; the motion according to $g C$ has no tendency to make the point g turn round the centre C. The motion of the point g parallel to $C S$ will be $\dfrac{S}{\overline{S g}^2} \times \dfrac{S C}{S g}$ or $\dfrac{S \times S C}{\overline{S g}^3}$; but the motion of the centre C according to $C S$ is $\dfrac{S}{\overline{S C}^2}$; therefore the relative motion of the point g, in regard to the centre C, is $\dfrac{S \times S C}{\overline{S g}^3} - \dfrac{S}{\overline{S C}^2}$ or $S \times S C \times$

$$\left(\dfrac{1}{\overline{S g}^3} - \dfrac{1}{\overline{S C}^3} \right).$$

The moment of the point g to turn about the centre C, will therefore be $S \times S C \times \left(\dfrac{1}{\overline{S g}^3} - \dfrac{1}{\overline{S C}^3} \right) \times g \times C I$,

the moment being the product of the motion by the mafs of the body, and by the arm of the leaver. And this moment caufes the point I to approach towards the fun, when it is pofitive, or when $S C$ is greater than $S g$; and caufes the point I to recede from the fun when it is negative, or when $S g$ is greater than $S C$.

It

It will be found, in like manner, that the moment of the point g' to turn about the centre C, is $S \times S\,C \times$

$$\left(\frac{1}{\overline{Sg'}^2} - \frac{1}{\overline{S\,C}^3} \right) \times g' \times C\,I';$$ or with regard to the point I which is on the oppofite fide of the centre to the point I', this moment, by changing the figns, will be, $S \times S C \times \left(\frac{1}{\overline{S\,C}^3} - \frac{1}{\overline{S\,g'}^3} \right) \times g' \times C\,I' = S \times S C \times$

$$\left(\frac{1}{\overline{S\,C}^3} - \frac{1}{\overline{S\,g'}^3} \right) \times g \times C\,I.$$ Therefore the moment of the two equal points g and g' to caufe the point I to turn about the centre C, is $S \times S\,C \times \left(\frac{1}{\overline{S g}^3} - \frac{1}{\overline{S C}^3} \right)$

$$\times g \times C I + S \times S C \times \left(\frac{1}{\overline{S\,C}^3} - \frac{1}{\overline{S g'}^3} \right) \times g \times C\,I =$$

$$g \times C\,I \times S \times S\,C \times \left(\frac{1}{\overline{S g}^3} - \frac{1}{\overline{S g'}^3} \right).$$

If the points γ and γ' be taken equally diftant from the points D and D', as the points g and g', it will likewife be found, that the moment of the equal points γ and γ' to caufe the point I to turn about the centre C, is $\gamma \times \overline{C\,I} \times S \times \overline{S\,C} \times \left(\frac{1}{\overline{S\gamma}^3} - \frac{1}{\overline{S\gamma'}^3} \right) = \gamma \times$

$$\overline{C\,I} \times S \times S\,C \times \left(\frac{1}{\overline{S g'}^3} - \frac{1}{\overline{S g'}^3} \right).$$

Therefore the moment of all the four points together, g, g', γ, γ', to caufe the point I to turn about the centre C, is $(g-\gamma) \times \overline{CI} \times S \times S\,C \times \left(\frac{1}{\overline{S g}^3} - \frac{1}{\overline{S g'}^3} \right)$,

which

which becomes $= 0$ when the point g can be taken $=$ γ, as may be done throughout the whole extent of the circle $P E P' E' P$; but if the zone $E P' E' P E F' E' F E$ be taken away from this circle, to have the true figure of the earth, we fhall have $g = \overline{gf} \times \overline{gr}$, and $\gamma =$ $- \overline{\gamma f} \times \overline{\gamma \varrho}$, taking the points g and γ for the elements of the zone, and $g - \gamma = \overline{gf} \times \overline{\gamma \varrho - gr}$, becaufe $\overline{\gamma f} = \overline{gf}$.

Let the lines $C S$ and $C P$, or $C E$, be called s and a, refpectively; the radius 1, or unity; the fine of the angle $S C P$, V; its cofine, or the fine of the angle $S C E$, u; $C K$, x; $g K$, y; $\overline{g m}$, z; $\overline{\gamma M}$, z'; $\overline{gf}$ or $\overline{\gamma f'}$, du; $P F$, $a a$; we fhall have $g n = a z$; $\gamma v = a z'$ (by the property of the ellipfe); alfo calling $\overline{g r}$, r; and $\gamma \varrho$, ϱ; and regarding the little triangle $g r n$ as a right-lined one, and fimilar to the triangle $g C m$, we fhall have $r = \dfrac{z}{a} \times a z$, and likewife $\varrho = \dfrac{z'}{a} \times a z'$.

By the proportion of fines to the fides of triangles, we fhall have $\gamma G = y V$: $C N = x u$. Finally, if we confider the point S, or the fun, as at an infinite diftance, we fhall have $S g = S K = S C - C K = s - x$, and $S g' = S K = S C + C K' = s + x$. Therefore $\dfrac{1}{\overline{Sg}^3} = \dfrac{1}{s^3 - 3 s^2 x + 3 s x x - x^3}$, and $\dfrac{1}{\overline{Sg'}^3}$ $= \dfrac{1}{s^3 + 3 s^2 x + 3 s x^2 + x^3}$. Therefore $\dfrac{1}{\overline{Sg}^3} - \dfrac{1}{\overline{Sg'}^3} =$

$6 s^2 x$

$$\frac{6s^2x + 2x^3}{(s-x)^3 \times (s+x)^3};$$ and, rejecting the infinitely small magnitudes, $= \frac{6x}{s^4}.$

Therefore the moment wherewith the four points g, g', γ, γ', cause the point I to turn, will, by substituting in the place of the letters their analytical values, be $(\varrho\, du - r\, du) \times S \times y \times \dfrac{6x}{s^3}$; or putting, instead of ϱ and r, the values above found, we shall have for the moment $\left(\dfrac{a}{a'}z'z' - \dfrac{a}{a}zz\right) \times du \times S \times y \times$

$$\frac{6x}{s^3} = \frac{6\,aS}{s^3 a} \times y\, x\, d\, u \times (z' + z) \times (z' - z). \quad \text{But}$$

$z' + z = \gamma\, M + g\, m = 2\ \overline{\gamma\, G} = 2\, y\, V$; and $z' - z = \gamma\, M - g\, m = 2\ \overline{G\, M} = 2\ \overline{C\, N} = 2\, x\, u.$ Therefore $(z' + z) \times (z' - z) = 4\, V\, u\, x\, y$; and the moment is $\dfrac{6aS}{s^3} \times 4\, V\, u \times x\, x\, y^2\, du = \dfrac{6aS}{s^3} \times 4 V u \times x\, x\, d\, x\, u.$

But in the circle $D\, g\, I$ we have $y\, d\, u = a\, d\, x$, and the moment is $\dfrac{6aS}{s^3} \times 4 V u \times x\, x\, y\, d\, x = \dfrac{6aS}{s^3} \times 4\, V\, u \times d\, x$

$\sqrt{aa - xx}.$ But the integral of $x^2\, d\, x\ \sqrt{aa x x}$ is $\frac{1}{4}$

$a\, a \int d\, x\ \sqrt{aa - xx} - \dfrac{x}{4}\, (a\, a - x\, x)^{\frac{3}{2}},$ and when

$x = a$, this integral is $\frac{1}{4}\, a\, a \int d\, x\ \sqrt{aa - xx}$, or, calling the ratio of the circumference to the diameter π, we shall have $\int d\, x\ \sqrt{aa - xx}$, or the area of the quadrant whereof a is the radius, $= \frac{1}{2}\, a\, \pi \times \frac{1}{2}\, a = \frac{1}{4}\, a^2\, \pi.$

Therefore the integral of $\dfrac{6a'S}{s^3} \times 4\, V\, u \times x\, x\, d\, x$

$\sqrt{aa}$

$\sqrt{aa-xx}$, or the fum of the moments of all the points which compofe the elliptic zone $E\ P'\ E'\ P\ E\ F\ E\ F\ E$ is $\dfrac{6aS}{s^3} \times \dfrac{\overline{V}\ u\ \pi\ a^4}{4}$.

If the terreftrial fpheroid be cut by any plane parallel to the plane $P\ E\ P'\ E'\ P$ of the circle of declination, *fig.* 1. $n°$ 2. the fection will be an ellipfe fimilar to the ellipfe $E\ F'\ E'\ F\ E$; and if the greater femiaxe $c'\ L$ of this ellipfe be called X, it will be found, as has been already in the ellipfe $E\ F'\ F\ F\ E$, that the fum of the moments of all the points of this ellipfe to turn about its centre C, is $\dfrac{6a\ S}{S^3} \times \dfrac{\overline{V}u_\pi X^4}{4}$.

It has not been taken into confideration, that here the centre S of the fun is a little below the plane of the fection, the line $c'\ S$ making an infenfible angle with the line $C\ S$.

Calling $C\ c'$, Υ; the moment of all the fections parallel to $E\ F'\ E'\ F\ E$ to turn about the axe $C\ c'$ will be the integral of $\dfrac{3\ a\ S}{S^3} \times \dfrac{\overline{V}\ u\ \pi\ X^4}{2} \times d\ \Upsilon$, or the integral of $\dfrac{3aS}{s^3} \times \dfrac{\overline{V}\ u\ \pi}{2} \times (aa\ \Upsilon\Upsilon)^2 \times d\ \Upsilon = \dfrac{3aS}{s^3} \times \dfrac{\overline{V}\ u\ \pi}{2} \times$ $(a^4 d\ \Upsilon - 2a^2\ \Upsilon^2\ d\ \Upsilon + \Upsilon^4\ d\ \Upsilon)$, and this integral is $\dfrac{3\ a S}{s^3} \times \dfrac{\overline{V}\ u\ \pi}{2} \times (a^4\ \Upsilon - \tfrac{2}{3}\ a^2\ \Upsilon^3 + \tfrac{1}{5}\ \Upsilon^5)$; and when $\Upsilon = a$, the integral is $\dfrac{3\ aS}{s^3} \times \dfrac{\overline{V}\ u\ \pi}{2} \times \tfrac{8}{15}\ a^5$. This is the fum of the moments of all the points of the hemifpheroid of the earth formed by the fection $E\ P'\ E'\ P\ E$ of the circle of declination, and the fum of the moments of all the points of the whole fphe-
roid

roid to turn about the axe $\mathcal{Q}\mathcal{Q}'$ is $\frac{3aS}{s^3} \times V u_\pi \times \frac{8}{15}a^5$.
Which was to be found.

COROLLARY.

2. If it were required to find the fum of the moments of all the points of a fingle crown $E L \mathcal{Q} E' \mathcal{Q}' E$, *fig.* 1. n° 2. placed at the equator, and detached from the earth, to turn about the axe $\mathcal{Q}\mathcal{Q}'$ of the circle of the fun's declination, the thicknefs of the crown being a a:

The moment of the point L to turn about c' is the fame as that of the point γ to turn about C; as is manifeft from the refolution of the motion according to $S L$ into two, $L\gamma$, and γS^*; and from the great diftance of the fun γS will not differ from $L S$; the triangle $L \gamma S$ being right-angled at γ.

By the preceding problem, if $c' L$, or $C \Upsilon$, be called X; and $C c'$, Υ; and all the other denominations of that problem be retained; we fhall have $\gamma x = X u$, and $C x = X V$.

And the moment of the point γ is $\gamma \times$

$$\gamma x \times S \times \overline{SC} \times \left(\frac{1}{\overline{S\gamma^3}} - \frac{1}{\overline{SC^3}}\right) = \gamma \times X u \times S \times \frac{3XV}{s^3}.$$

And putting inftead of γ, the element of the crown, which is $a a \times d u$, calling, in this cafe, $d u$ the element of the circumference $E L \mathcal{Q} E'$, the moment of an element of the crown will be a$a \times \frac{3S}{s^3} \times V u \times$

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* The lines $S L$ and γS are not drawn in the *fig.* to avoid confufion.

$X X d u$

$XXdu$. But in the circle $E\,Q\,E'\,E$ we have, $X\,d\,u$ $= a\,d\,\Upsilon$; therefore the moment of an element of the crown is $aa \times \dfrac{3S}{s^3} \times V\,u \times a \times d\,\Upsilon = \dfrac{3S}{s^3} \times a a^2\,V\,u \times$ $d\,\Upsilon\sqrt{\overline{aa - \Upsilon\Upsilon}}$; and the sum of the moments of the quadrant $E\,L\,Q$ of the crown is $\dfrac{3S}{s^3} \times a a^2\,V\,u \times$ $\tfrac{1}{2}\,a\,\pi \times \tfrac{1}{2}\,a = \dfrac{3S}{s^3} \times a\,V\,u \times \tfrac{1}{4}\,\pi\,a^4$. And every other quadrant of the crown having an equal motion, the sum of the moments of all the points of the crown, to turn, in vertue of the folar action, about the axe $Q\,Q'$, is $\dfrac{3S}{s^3} \times a\,V\,u \times \pi\,a^4$.

PROBLEM II.

3. *To find the sum of the moments of all the points of the terreftrial spheroid turning about the axe* $Q\,Q'$, *the motion of the point* E *being given, and* $= \mu$.

We muft firft feek the sum of the moments of all the points of the ellipfe $E\,F'\,E'\,F\,E$, *fig.* 1. n° 2. turning about its centre C; and to have the sum of the moments of all the points of this ellipfe, we fhall firft find the sum of the moments of all the points of the circle $E\,P'\,E'\,P\,E$, and then we fhall fubduct therefrom the sum of the moments of the whole elliptic zone $E\,P'\,E'\,P\,E\,F'\,E'\,F\,E$.

The motion of any point of the circle $E\,P'\,E'\,P\,E\,C$, placed at the diftance x from the centre C, will be $\dfrac{x}{a}\,\mu$; the motions being here in the fame ratio as the di-

4   ftances

ſtances from the centre, ſince all the points com-plete their revolution in the ſame time; and the mo-ment of all the points which compoſe the circum-ference $2 \pi x$ will be $2 \pi x \times \frac{x}{a} \mu \times x$; the moment being produced from the maſs by the motion, and by the arm of the leaver.

Therefore the motion of all the concentric circum-ferences which compoſe the circle, is $\int \frac{2 \pi \mu}{a} \times x^3 \, d x$

$= \frac{1}{2} \frac{\pi \mu}{a} \times x^4 = \frac{1}{2} \frac{\pi \mu}{a} \times a^4$, when $x = a$, as for the circle $E \, P' E' P \, E$.

To have the ſum of the moments of all the points of the elliptical zone $E \, P' \, E' \, P \, E \, F' \, E' \, F \, E$, all the points of the zone may be conſidered as placed at the diſtance a, from the centre C, and having, conſe-quently, the ſame motion as the point E, becauſe the greateſt thickneſs $P \, F$ of the zone is very ſmall.

The ſum of the moments of the zone will be equal to the quantity of matter multiplied by the motion μ, and by the arm of the leaver a, or $2 a \pi \times \frac{1}{2} a u \times \mu \times a = \pi \mu a a^3$.

Therefore the ſum of the moments of all the points of the ellipſe $E \, F \, E' \, F \, E$ turning about its centre C, the motion of the point E being μ, will be

$\frac{1}{2} \frac{\pi \mu}{a} \times a^4 - \pi \mu a a^3$.

It will likewiſe be found generally, that the ſum of the moments of a ſimilar ellipſe, of which $C' L$, or X, is half the greater axe, and which turns about its centre C', is $\frac{1}{2} \frac{\pi \mu}{a} \times X^4 - \frac{\pi \mu a}{a} \times X^4$.

Therefore

Therefore the fum of all the fimilar ellipfes which compofe the terreftrial hemifpheroid, is $\int \frac{1 - 2a}{2a} \times$

$\pi \mu \times X^4 \, d \Upsilon = \frac{1 - 2a}{2a} \times \pi \mu \times \int (aa - \Upsilon\Upsilon)^2 \times d\Upsilon$, which, as has been feen at the end of *Prob.* I. when, after the integration Υ has been made $= a$, will be $\frac{1 - 2a}{2a} \pi \mu \times \frac{8}{15} a^5$, and multiplying by 2, the fum of all the points of the terreftrial fpheroid will be $(1 - 2a) \times \pi \mu \times \frac{8}{15} a^4$. Which was to be found.

COROLLARY.

4. If you would find the fum of the moments of all the points of the ring $E \mathcal{Q} E' \mathcal{Q}' B$, *fig.* 1. *n*. 2. turning about the axe $\mathcal{Q} \mathcal{Q}'$:

The moment of an element L of the crown placed at the diftance X from the axe of rotation $\mathcal{Q}\mathcal{Q}'$, is a $a \, d u \times \frac{\mu X}{a} \times X$, which is the product of the quantity of matter by the motion, and by the arm of the leaver. But in the circle $E \mathcal{Q} E'$ E we have, $d u \frac{a d \Upsilon}{X}$;

therefore the moment of an element of the crown is a $a \mu \times X d \Upsilon$, whofe integral, which is a $a \mu \times a a \pi$, gives the fum of the moments of all the points of the crown. Which was to be found.

PRO-

PROBLEM III.

5. *Having given, the moment wherewith the earth turns about the axe* Q Q' *of the circle of the sun's declination, To find the motion of the pole* P, *or, which is the same, of the point* E *in the plane of the circle of the sun's declination.*

By *Prob.* I. the moment wherewith the earth turns about the axe $C \, \mathcal{Q}$, by the action of the sun, is $\frac{3 \, S}{s^3}$ $\times$ a $\pi \, v \, u \times \frac{8}{15} \, a^5$.

By *Prob.* II. the moment wherewith the earth turns about the same axe $C \, \mathcal{Q}$, the velocity of the point P, or of the point E, being μ, is $(1-2a) \times \pi \, \mu \times \frac{8}{15} \, a^4$.

We shall have then $\frac{3 \, S}{s^3}$ a $\pi \, v \, u \times \frac{8}{15} \, a^5 = (1-2a)$ $\times \pi \, \mu \times \frac{8}{15} \, a^4$; whence we get $\mu = \dfrac{3 \, S \, a \, a \, v \, u}{s^3 \times (1 - 2a)}$. And this is the motion of the pole P, or of the point E, about the centre C. Which was to be found.

COROLLARY.

6. In like manner, the moment wherewith the crown $E \, \mathcal{Q} \, E' \, \mathcal{Q}' \, E$ turns about the axe $\mathcal{Q} \, \mathcal{Q}'$ being given, the motion of the point E may be known. For,

By the *Coroll.* of *Prob.* I. the moment of this crown, turning about the axe $C \mathcal{Q}$ by the action of the sun, is $\frac{3 \, S}{s^3} \times$ a $\pi \, v \, u \times a^4$.

By

By the *Coroll.* of *Prob.* II. the moment of this crown is alſo a $a \mu \times a^2 \pi$.

We have then $\frac{3S}{s^3} \times a \pi \, v \, u \times a^4 = a \, a \, \mu \times a^2 \pi$;

whence we get $\mu = \frac{3S}{s^3} \times a \, v \, u$. Which was to be found.

Remark. It is proper to obſerve, that the motion of the point E of the crown is the ſame, whether the crown be entire, or there be no more of it but the point E.

For, by *Prob.* I. the motion of any point g, parallel to CS, *fig.* 1. n° 2. and reſpectively to the centre C, is $S \times \overline{SC} \times \left(\frac{1}{\overline{Sg}^3} - \frac{1}{\overline{SC}^3} \right)$, which is $= S \times \frac{3x}{S^3}$; and when the point g becomes the point E, we have $x = a \, v$, v being the coſine of the angle DCE. Therefore the reſpective motion of the point E parallel to the motion CS of the centre C, is $S \times \frac{3 \, a \, v}{S^3}$; and reſolving this motion into two, the one according to EC, and the other perpendicular to EC, this latter will be $= S \times \frac{3 \, a \, v}{s^3} \times \frac{a \, u}{a} = \frac{3S}{s^3} a \, v \, u$; being the ſame as that which was found for the point E when it is united to the crown.

This ſhews that the motion of the other points of the crown produce no alteration in the motion of the point E, which is owing to this; that all the points of the crown have the ſame angular motion about the axe $C\mathcal{Q}$.

PRO-

PROBLEM IV.

7. *To find at every inftant the variation of the place of the pole of the earth.*

The earth has two motions of rotation, the one about the axe of the equator, which is likewife the earth's axis; and the other about the axe $C\mathcal{Q}$ of the circle of the fun's declination; this motion is caufed by the fun's action on the redundant matter about the equator.

The point E, *fig.* 1. n° 2. which is the interfection of the circumference $E\mathcal{Q}E'\mathcal{Q}'E$ of the equator and the circle $P E P' E' P$ of the fun's declination, will therefore have at every inftant two motions, whofe directions are perpendicular to each other.

The former of thefe two motions, which is the diurnal motion, is uniform; and if we call it m, the fpace run through in an inftant, $d\,t$, is $m\,d\,t$; the motion being always equal to the fpace divided by the time, or the fpace equal to the product of the time. Alfo let $E\,e$ be that fpace.

The fecond motion of the point E which is performed in the circumference $E\,P\,E'\,P'\,E$ of the circle of the fun's declination, and arifes from the action of the fun, is continually accelerated, from the continual application of the fun's action; and if we call the initial motion μ, at the firft term of the inftant, $d\,t$, or the increment of the motion, the motion at the end of the inftant $d\,t$, is $\mu\,d\,t$; and the fpace gone thro' uniformly by the motion during the inftant $d\,t$, is $\mu\,d\,t \times d\,t$, or $\mu\,d\,t^2$. Let $E\,\epsilon$ be that fpace.

The

The point E, in confequence of thefe two motions together, $E\,e$ and $E\,\varepsilon$, circulates neither in the circumference $E\,\mathcal{Q}\,E'\,\mathcal{Q}'\,E$ of the equator, nor in the circumference $E\,P\,E'\,P'\,E$ of the fun's declination. But if we form the rectangle $E\,e\,e'\,\varepsilon\,E$, the diagonal $E\,e'$ will be the elementary arc of the circumference $E\,e'\,q\,E'\,E$ wherein the point E will circulate; and the angle $e\,E\,e'$, equal to the angle $\mathcal{Q}\,c\,q$, equal to the angle $P'\,C\,p'$, will be the angle whereby the pole P' is elevated, and the pole P depreffed, below the circle of declination, by moving in the circumference $P'\,p'\,\mathcal{Q}\,q\,P\,p\,\mathcal{P}'$, whofe plane is perpendicular to the plane of delination; and the point p' is the true place of the pole at the end of the inftant $d\,t$, and the fmall arc $P'\,p'$ expreffes the inftantaneous variation of the place of the earth's pole. Which was to be found.

<center>CŎROLLARY I.</center>

8. The fimilar fectors $e\,E\,e'$, $\mathcal{Q}\,C\,q$, or $P'\,C\,p'$, whofe three fides of the one are each parallel to the three fides of the other, give this proportion, $E\,e\,:$

$e\,e'$ or $E\,\varepsilon\,::\,C\,P'\,:\,P'\,p' = \dfrac{E\,\varepsilon}{E\,e} \times C\,P' =$ (by the

preceding $Prob.$) to $\dfrac{\mu\,d\,t^2}{m\,d\,t} \times C\,P' = \dfrac{\mu\,d\,t}{m} \times a.$ But

by $Problem$ III. $\mu = \dfrac{3\,S\,a\,a\,\nu\,u}{s^3\,(1-2a)}$; therefore $P'\,p' =$

$\dfrac{3\,S\,a\,a\,\nu\,u}{s^3 \times m \times (1-2a)} \times a\,d\,t = A\,\nu\,u\,a\,d\,t$, making A

$= \dfrac{3\,S\,a\,a}{s^3 \times m \times (1-2a)}.$

COROLLARY II.

9. If, instead of the terrestrial spheroid, we supposed only a single ring, it would be found in like manner, that the pole P of such ring would shift its place during the instant $d\,t$, by running through the arc $\mathcal{P}'\,p' = \dfrac{E\,i}{E\,e} \times C\,P' = \dfrac{\mu\,d\,t}{m} \times a$. But then we should take the value of μ as in *Coroll.* of *Prob.* III. which is $\dfrac{3\,S}{s^3} \times a\,\nu\,u$. Therefore we shall have $P'\,p' = \dfrac{\mu\,d\,t}{m} \times a = \dfrac{3\,S\,a}{s^3 \times m} \times \nu\,u\,a\,d\,t$.

To apply, then, the *formula* of the preced. *Coroll.* it will suffice to make $\dfrac{a}{1 - 2a}$ in that *Coroll.* $= 1$. for then the *formula* of *Coroll.* I. becomes the same with that just now found in this *Coroll.*

This will be of use hereafter for applying the *formulæ* of the motion of the nodes of the terrestrial equator to the motion of the nodes of a moon or satellite of a planet.

COROLLARY III.

10. It should be well observed in the preceding *Coroll.* that the direction $P'\,p'$ of the pole P' is ever parallel to the direction $E\,e$ of the point E of the equator, since they are both perpendicular to the plane of the circle of the sun's declination, and the direction $P'\,p'$ goes on the same side as the direction $E\,e$ of the point E, which is placed on the same side of the
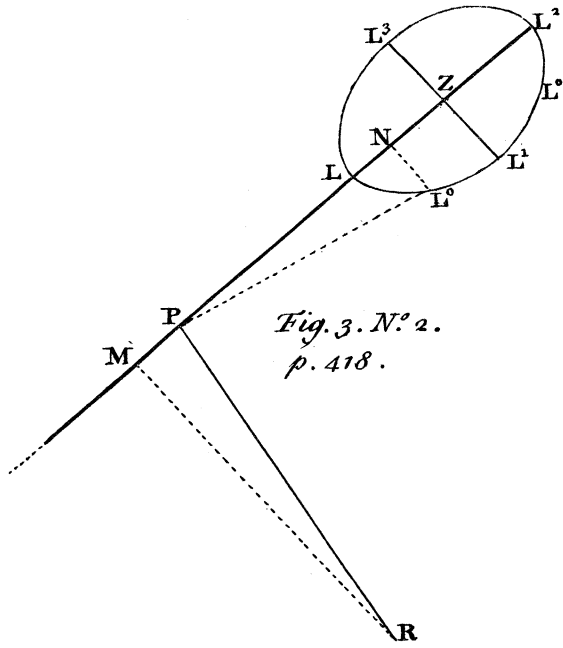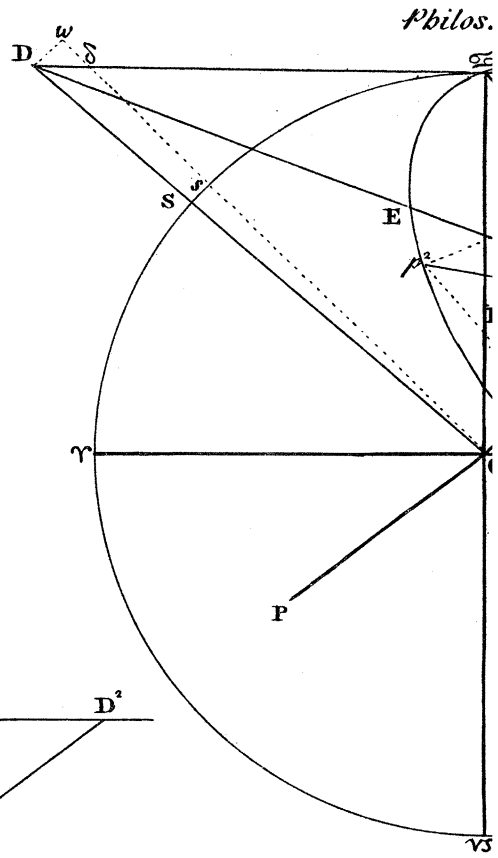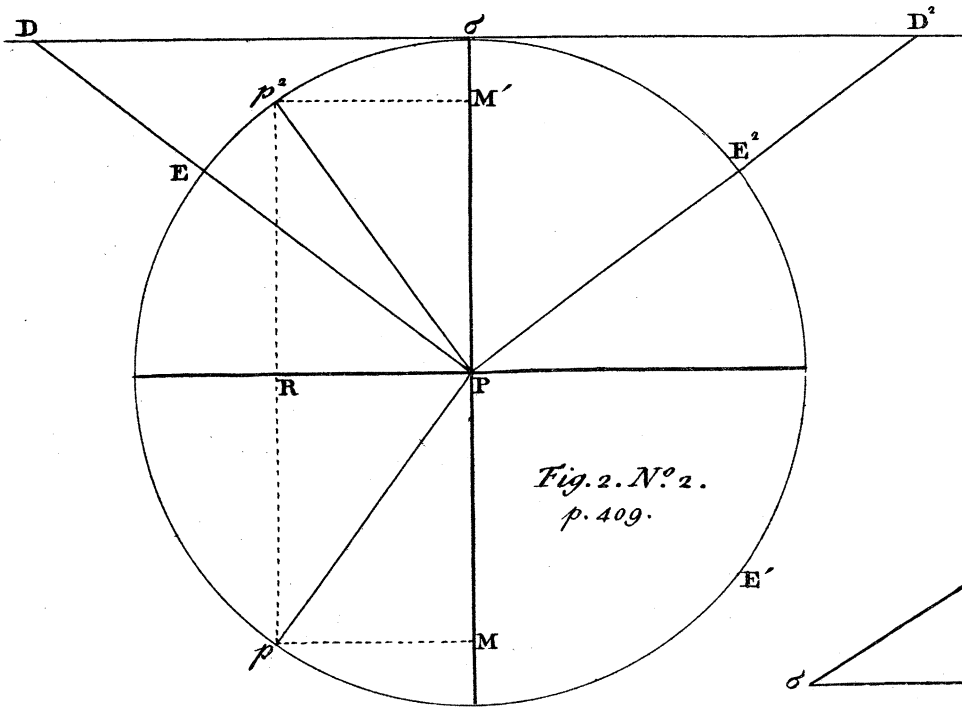
Fig. 3. N.º 2.
p. 418.
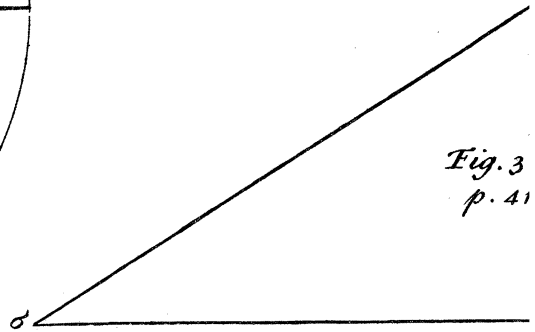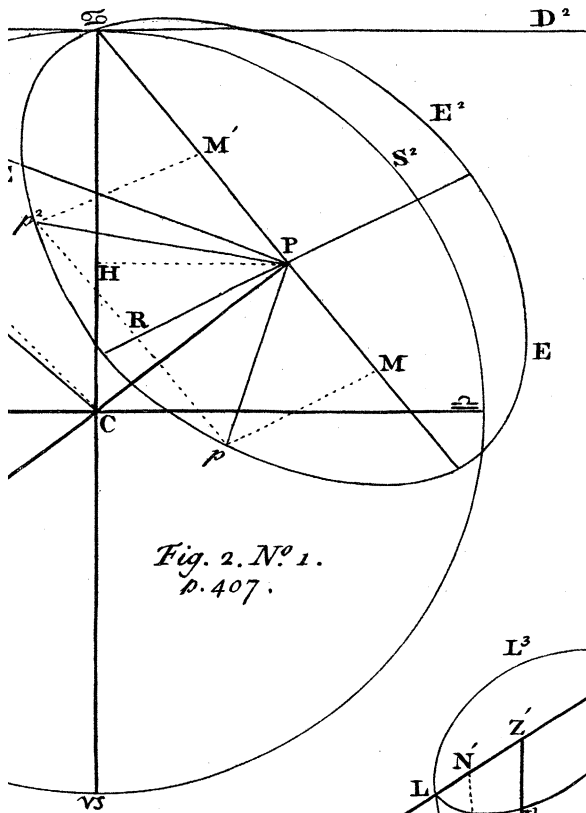
Fig. 2. N.º 2.
p. 409.
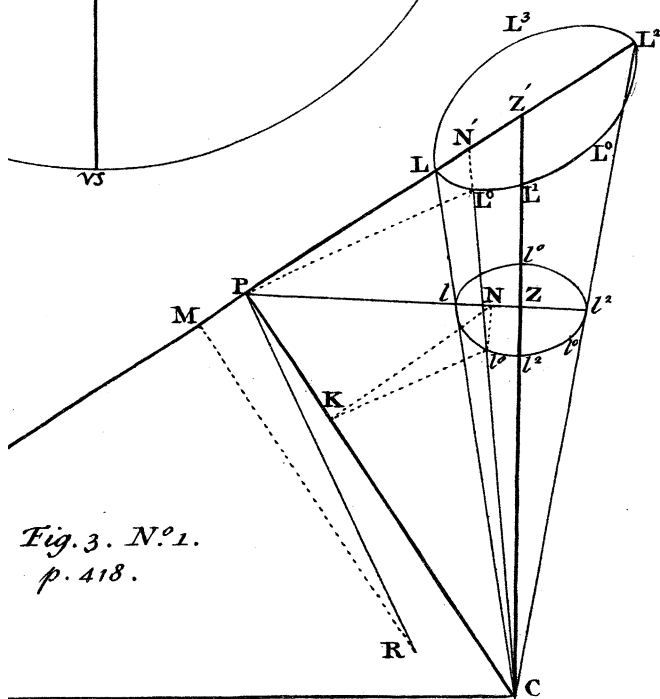
Fig. 3
p. 41

Fig. 2. N.º 1.
p. 407.

Fig. 3. N.º 1.
p. 418.

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the pole *P'*, with refpect to the line *S C D'*, and the
direction of the oppofite pole *P*, goes on the fame
fide as the direction of the point *E* of the equator
which is placed on the fame fide of the pole *P* with
refpect to the line *S C D'*. This fhould be carefully
attended to in regard to the following *Problem*.

PROBLEM V.

11. *To determine at every inftant the motion of the
pole of the earth with refpect to the ecliptic.*

P P', *fig.* 2. *n°* 1. & *n°* 2*. is the earth's axe;
♈ ♋ ♎ ♑ reprefents the ecliptic; ♋ *E p E* ♋ is the
tangent plane to the pole *P*, and parallel to the equa-
tor; the line *D* ♋ *D²* is the interfection of this plane
with that of the ecliptic, and is tangential at the com-
mon point ♋ to the circles ♋ *E' p E* ♋ and ♈ ♋ ♎ ♑.
The angle *P C* ♋ is the angle of inclination of the
earth's axe to the plane of the ecliptic, being like-
wife the folfticial angle. The earth, by its diurnal
motion, turns about the axe *P' P* according to the
order of ♋ *E' p E* ♋; and the fun, by the annual
motion in the ecliptic, moves according to the order
of ♈ ♋ ♎ ♑.

Let *S* be any place of the fun in the ecliptic. By
Art. 10. the motion of the pole *P* making the acute
angle *P C S*, will be according to *P p* perpendicular
to the plane *P C S D P*, and confequently to the line
P D.

---

* *Note*, That the *fig. n°* 2. is only to reprefent the circle
♋ *E p* ♋, which could not be reprefented but by projection in
fig. n° 1.

5   Forming,

Forming, therefore, the rectangle $P M p R$, the motion of the pole according to $P p$ will be refolved into two, according to $P R$ and $P M$.

The motion of the pole according to $P R$ is parallel to the plane of the ecliptic, and to the line of the nodes ♈ C ♎. This motion tends to move the pole P about the axe of the ecliptic, and in a parallel to the ecliptic, and caufes the equinoctial point ♈ to recede, and the point ♏, and the node ♎, to advance towards the fun; which produces a preceffion of the equinox equal to the angular motion of the pole P about the axe of the ecliptic; for it is indifferent whether the pole P of the earth moves about the axe of the ecliptic, whilft the plane ♈ ♋ ♎ ♑ remains fix'd; or the fame be fuppofed to be fix'd, and the plane of the ecliptic ♈ ♋ ♎ ♑ to turn about its centre C, and the axe of the ecliptic.

If we take another place S^2 of the fun at the fame diftance from the folftice ♋, but on the fide contrary to S, the direction of the pole P will be, by *Art.* 10. according to $P p^2$, perpendicular to the plane $P C S^2 D^2 P$, and confequently to the line $P D^2$; and the fun's declination being the fame in D^2 as in D, the motion $P p^2$ will be equal to the motion $P p$.

Refolving the motion according to $P p^2$ into two, according to $P K$ and $P M'$, the motion according to $P R$ produces the fame preceffion, and with the fame fign, or the fame direction, as when the fun was on the other fide of the folftice ♋, and in D; and the preceffion is ever the fame at every quarter revolution of the fun.

The motion according to $P M'$ caufes the fame variation in the inclination of the earth's axe to the
ecliptic,

ecliptic, as when the fun was on the other fide in *S*, but with a contrary fign, or direction; and the fum of the lines *P M'*, from the folftice ♋ to ♎, will be the fame as the fum of the lines *P M*, from ♈ to ♋, but with a contrary fign: fo that the angle of inclination of the earth's axe to the plane of the ecliptic is continually increafing from ♈ to ♋, and then decreafing from ♋ to ♎; and as the diminution is equal to the increafe, the inclination becomes again the fame in ♎ as it was in ♈, having undergone an ofcillation whofe period is fix months.

Whilft the fun paffes from ♎ to ♈, there are the fame motions of the pole, as whilft it was paffing from ♋ to ♎.

Thus is the motion of the pole determined at every inftant, with refpect to the ecliptic. Which was to be done.

Problem VI.

12. *To find the quantity of the preceffion, or, which is the fame, the arc of the parallel to the ecliptic run through by the pole of the earth* P *in the fpace of time the fun is paffing from the equinox to the folftice, or from the folftice to the equinox; which may be taken for a quarter of the time of the fun's revolution about the earth; the angle comprehended between the equinox and the enfuing folftice differing but by an infinitely fmall quantity from 90 degrees.*

Calling, as in *Prob.* I. the femi-axe *C P*, *a*; *fig.* 2. n° 1. & n° 2. the fine of the angle *P C S*, or *P C D*, *V*; and its cofine, or the fine of the fun's declination, *u*:

F f f   Let

Let $P\text{\it{\&}}$, be b; $C\text{\it{\&}}, r$; we shall have $CD = \dfrac{a}{u}$;

$PD = \dfrac{a}{u}V$; the difference Ss of the arc ΥS, which is

the sun's distance from the node Υ, will be $\dfrac{rr \times d\overline{CD}}{\overline{CD} \times \overline{\text{\it{\&}}D}}$,

as is found by these two proportions, $r : \overline{Ss} :: CD$

$: D\omega = \dfrac{CD \times \overline{Ss}}{r}$, and $D\omega = \dfrac{CD \times \overline{Ss}}{r} : \delta\omega =$

$d\overline{CD} : \overline{C\text{\it{\&}}} = r : \overline{\text{\it{\&}}D}$; whence we have $\overline{CD} \times \overline{Ss}$

$\times \overline{\text{\it{\&}}D} = rr \times d\overline{CD}$, and $\overline{Ss} = \dfrac{rr \times d\overline{CD}}{\overline{CD} \times \overline{\text{\it{\&}}D}} =$

$\dfrac{r\,d\,u}{\sqrt{\overline{aa - rr}}}$ (putting instead of $\overline{CD}$ and $\overline{SD}$, their
$\phantom{\dfrac{r\,d\,u}{\sqrt{\overline{aa}}}}\overline{uu}$

values $\dfrac{a}{u}$ and $\sqrt{\overline{\dfrac{aa}{uu} - rr}}$) $=$ to the space run through

by the sun S in the ecliptic in an instant, $= 2\pi r \times \dfrac{dt}{T}$,

calling T the time the sun employs in running thro'
the whole circumference $2\pi r$.

Whence we have $dt = \dfrac{T}{2\pi r} \times \dfrac{r\,d\,u}{\sqrt{\overline{aa - uu}}} = \dfrac{T}{2\pi}$
$\phantom{Whence we have dt = \dfrac{T}{2\pi r} \times \dfrac{r\,d\,u}{\sqrt{aa}}}\overline{rr}$

$\times \dfrac{d\,u}{\sqrt{\overline{aa - uu}}}$.
$\phantom{\times \dfrac{d\,u}{\sqrt{aa}}}\overline{rr}$

By *Art.* 8. we have $a\,P\,p = A\,V\,u \times a\,d\,t$; therefore, by *Prob.* V. the inftantaneous preceffion will be

$$A\,V\,u \times a\,d\,t \times \frac{\overline{PR}}{\overline{Pp}}, \text{ or } A\,V\,u \times a\,d\,t \times \frac{\overline{P\,\circleddash}}{\overline{P\,D}},$$

the triangles $\mathcal{P}\,p\,r$, $\mathcal{P}\,D\,\circleddash$, being fimilar, fince the three fides of the one are perpendicular to the three fides of the other. Putting, therefore, inftead of $\overline{P\,D}$, $\overline{P\,\circleddash}$, and $d\,t$, their values above found, we fhall have the

$$\text{inftantaneous preceffion} = A\,V\,u \times \frac{a\,b}{\dfrac{a\,V}{u}} \times \frac{T}{2\,\pi} \sqrt{\frac{aa - uu}{rr}}$$

$$= A\,b \times \frac{T}{2\,\pi} \times \frac{u\,u\,d\,u}{\sqrt{\dfrac{aa - uu}{rr}}}. \quad \text{But } \int \frac{u\,u\,d\,u}{\sqrt{\dfrac{aa - uu}{rr}}} =$$

$$u \sqrt{\frac{aa - uu}{rr}} + \int du \sqrt{\frac{aa - uu}{rr}}, \text{ or fimply} = \int \frac{u\,u\,d\,u}{\sqrt{\dfrac{aa - uu}{rr}}}$$

when $u = \dfrac{a}{r}$, as it happens when the fun is in the folftice $\circleddash$, and then $\int d\,u \sqrt{\dfrac{aa - uu}{rr}}$, which is the area

of a quadrant of a circle whofe radius is $\dfrac{a}{r}$, will be

$$\frac{a\,\pi}{2\,r} \times \frac{a}{2\,r} = \frac{aa\,\pi}{4\,rr}.$$

Therefore the preceffion, whilft the fun is paffing from Υ to $\circleddash$, is $A\,b \times \dfrac{T}{2\,\pi} \times \dfrac{aa\,\pi}{4\,rr} = A\,b \times \dfrac{T}{2} \times \dfrac{aa}{4\,rr}$;

and

and calling CH, which is $=$ the radius of the circle parallel to the ecliptic which the pole P runs thro',

g; we fhall have $g = \dfrac{aa}{r}$, and the preceffion $Ab \times$

$\dfrac{T}{2} \times \dfrac{aa}{4rr}$ will be likewife $Ab \times \dfrac{T}{2} \times \dfrac{g}{4r}$. Which was to be found.

And this alfo is the quantity of the preceffion during the time the fun paffes from the folftice to the equinox, or generally, during the time of each quadrant of the fun's revolution; as was feen in *Prob.* V.

Remark.

13. The difference of the fun's diftance $S\Upsilon$ from the equinoctial point, is equal to the arc the fun goes through in the ecliptic, when the node Υ is fix'd, as was fuppos'd in the foregoing *Prob.* but when the node Υ moves in the ecliptic the contrary way to the fun's motion, the difference of the fun's diftance from the node is equal to the arc gone through by the fun, increafed by the arc gone through by the node Υ.

Therefore always taking Ss, *fig.* 2. n° 1. for the difference of the fun's diftance from the node, we fhall have $\overline{Ss}$ equal to the arc defcribed by the fun, more by the arc defcribed by the node.

But, by the former *Prob.* we have feen, that the arc run through by the pole P in an inftant dt, and in a parallel to the ecliptic, whereof g is the radius, is

$$AVu \times a\,dt \times \dfrac{\overline{P\,\maltese}}{PD} = AVu \times \dfrac{ab}{\dfrac{aV}{u}} \times dt = Abu\,u\,dt.$$

The arc run through by the node Υ will be then
$$Abu$$

$A\,b\,u\,u\,d\,t \times \dfrac{r}{s}$, fince thefe arcs, which meafure equal angles, are as their radii g and r.

We fhall have, therefore, in the above *Problem*,

$\overline{S\,s}$, or $\dfrac{r\,d\,u}{\sqrt{\dfrac{aa-uu}{rr}}}$, $= 2\,\pi\,r \times \dfrac{d\,t}{T} + A\,b\,u\,u \times \dfrac{r}{g}\,d\,t$;

whence we get $d\,t = \dfrac{T\,d\,u}{2\,\pi\,\sqrt{\dfrac{aa-uu}{rr}}} \times \dfrac{1}{1 + A\,b\,u\,u \times \dfrac{T}{2\,\pi\,g}}$.

But $\dfrac{1}{1 + A\,b\,u\,u \times \dfrac{T}{2\,\pi\,g}} = 1 - A\,b\,u\,u \times \dfrac{T}{2\,\pi\,g} +, \&c.$

where the other terms may be neglected which contain the powers of A, becaufe A is a very fmall fraction.

We fhall have, therefore, $d\,t = \dfrac{T\,d\,u}{2\,\pi\,\sqrt{\dfrac{aa-uu}{rr}}} \times$

$\left(1 - A\,b\,u\,u \times \dfrac{T}{2\,\pi\,g}\right)$; and the inftantaneous pre-ceffion will be $A\,b \times \dfrac{T}{2\,\pi} \times \dfrac{u\,u\,d\,u}{\sqrt{\dfrac{aa-uu}{rr}}} \times \left(1 - A\,b\,u\,u \times \right.$

$\left. \dfrac{T}{2\,\pi\,g}\right)$ whofe integral for the firft term, in the folfti-cial point, where $u = \dfrac{a}{r}$, is $A\,b \times \dfrac{T}{2\,\pi} \times \dfrac{aa\pi}{4\,rr}$, as was feen in the preceding *Problem*. But the integral of

$u^4\,d\,u$

$$\frac{u^4\,d\,u}{\sqrt{\dfrac{aa-uu}{rr}}} = -\,u^3\sqrt{\dfrac{aa-uu}{rr}} - \tfrac{3}{4}\,u \times \left(\dfrac{aa}{rr}-uu\right)$$

$$+\,^3\frac{aa}{rr}\int d\,u\sqrt{\dfrac{aa-uu}{rr}},\ \text{or fimply,} = \tfrac{3}{4}\frac{aa}{rr}\int d\,u$$

$\sqrt{\dfrac{aa-uu}{rr}}$, when we fuppofe, after the integration,

$\dfrac{a}{r} = u$; and $= \tfrac{3}{4}\dfrac{aa}{rr}\times\dfrac{a\pi}{2\,r} = \tfrac{3}{4}\dfrac{aa}{rr}\times\dfrac{a\,a\,\pi}{4\,r\,r}$. Therefore the preceffion from the fun's departure from the equinox till his arrival at the folftice, is $A\,b\times\dfrac{T}{2}\times\dfrac{a\,a}{4\,r\,r}\times$

$\left(1-A\,b\times\dfrac{T}{2\pi g}\times\tfrac{3}{4}\dfrac{aa}{rr}\right) = \left(g\ \text{being} = \dfrac{aa}{r}\right)A\,b\times$

$\dfrac{T}{2r}\times\dfrac{g}{4}\times\left(1-A\,b\times\dfrac{T}{2\pi r}\times\tfrac{3}{4}\right)$.

When the motion of the node is not very fenfible, it will fuffice to make ufe of the *formula* of the preceding *Problem*, which is the firft term of this : But when that motion is fenfible, which is the cafe of the moon's node, when it is to be determined for a quarter of the revolution of the fun, it will be requifite to employ this *formula*, which contains two terms of the feries. Nay, it would be very eafy to take a greater number of terms, fo as to neglect nothing ; but this would be abfolutely ufelefs, becaufe of the quick converging of the feries.

Problem VII.

14. *To find the alteration of the inclination of the earth's axe to the ecliptic, during the sun's passage from the equinox to the solstice, or from the solstice to the next equinox; which is the same, only affected with a contrary sign, as has been seen in* Prob. V.

By *Art.* 8. we have $aPp = AVu \times a\,dt$; and by *Prob.* V. the instantaneous variation of the inclination of the earth's axe to the ecliptic is $AVu \times a\,dt \times$

$\dfrac{PM}{Pp} = AVu \times a\,dt \times \dfrac{\overline{\text{s}\,D}}{\overline{PD}}$, because of the similar

triangles $Pp\,M$, $PD\,\text{s}$, $= $ (taking the denominations

of *Prob.* VI.) $AVu \times a \times \dfrac{T}{2\pi} \times \dfrac{r \times d\overline{CD}}{\overline{CD} \times \overline{\text{s}\,D}} \times \dfrac{\overline{\text{s}\,D}}{\overline{PD}} = AVu \times$

$\dfrac{aT}{2\pi} \times \dfrac{r\,d\overline{CD}}{\overline{CD} \times \overline{PD}} = AVu \times \dfrac{aT}{2\pi} \times \dfrac{r \times \dfrac{du}{uu}}{\dfrac{ar}{u} \times \dfrac{a}{u}} = AV \times \dfrac{T}{2\pi}$

$\times u\,du$, whose integral is $\frac{1}{2}Ar \times \dfrac{T}{2\pi} \times uu$; and when

the sun is at the point s, we have $1 : u :: r : a$, and $a = ru$; therefore the preceding integral is then

$\frac{1}{2}A \times \dfrac{T}{2\pi r} \times aa$. Which was to be found.

Remark.

15. When the motion of the node is not insensible, you are to take the value of dt of the *Remark* that

follows

follows *Art.* 12. which gives $d\,t = \dfrac{T}{2\,\pi} \times \dfrac{r \times d\,\overline{C\,D}}{\overline{C\,D} \times \text{\ss}\,D} \times$

$(1 - A\,b \times \dfrac{T}{2\,\pi\,g} \times u\,u)$, and the inſtantaneous varia-

tion of the inclination of the earth's axe to the ecliptic

will be $A\,r \times \dfrac{T}{2\,\pi} \times u\,d\,u \times (1 - A\,b \times \dfrac{T}{2\,\pi\,g} \times u\,u)$,

whoſe integral, when the ſun arrives at the ſolſtice,

where we have $r\,u = a$, becomes $\frac{1}{2}\,A\,a\,a \times \dfrac{T}{2\,\pi\,r} -$

$\frac{1}{4}\dfrac{A\cdot a^{4}}{r^{2}} \times \dfrac{T}{2\,\pi\,r} \times A\,b \times \dfrac{T}{2\,\pi\,g} = \left(\text{becauſe } g = \dfrac{a\,a}{r}\right)$

$\frac{1}{2}\,A\,a\,a \times \dfrac{T}{2\,\pi\,r} \times \left(1 - \frac{1}{2}\,A\,b \times \dfrac{T}{2\,\pi\,r}\right)$; and this is

the *formula* to be made uſe of when the motion of
the nodes is ſenſible.

S E C T   II.

Of the motion of the pole of the equator of
the earth cauſed by the aďion of the moon.

16. Whatever has been ſaid in *Seď*. I. of the pre-
ceſſion, and of the oſcillation of the earth's axe on
the plane of the ecliptic, through the aďion of the
ſun, is equally applicable to the aďion of the moon;
it being requiſite only to ſubſtitute every-where
the moon inſtead of the ſun; L inſtead of S, and
λ inſtead of s; the time which the moon em-
ploys in completing her revolution about the earth,
inſtead of the time the ſun employs in performing

his revolution about the earth, or M inftead of T:
Laftly, the lunar orbit, inftead of the ecliptic. Like-
wife, inftead of A, which (*Art.* 8.) was equal to
$\frac{3\,S}{s^3 m}$, we muft put $B = \frac{3\,L}{\lambda^3 m} \times \frac{a\,a}{1-2a}$.

17. We fhall find then, by *Art.* 12. that the arc
defcribed by the pole P of the earth parallel to the
plane of the lunar orbit, whilft the moon makes one
quarter of a revolution about the earth, is $B\,b \times \frac{g}{2r} \times \frac{M}{4}$,
which gives the arc defcribed by the mean motion in
an inftant, $d\,t$, equal to $B\,b \times \frac{g}{2r} \times d\,t = B\,b \times \frac{a\,a}{2\,r\,r}$
$\times d\,t$; putting, inftead of g, its value $\frac{a\,a}{r}$.

18. We fhall find alfo, by *Art.* 14. that the varia-
tion of the inclination of the earth's axe to the plane
of the lunar orbit, when the moon is at its point of
ftation 90 degrees diftant from the node of the equa-
tor with the lunar orbit, is $\frac{1}{2} B \times \frac{M}{2\pi r} \times a\,a$, whereby
the angle of inclination of the earth's axe to the plane
of the lunar orbit is increafed when the moon is at its
point of ftation.

19. We therefore fhould have nothing to add, if
the plane of the lunar orbit were the fame as that of
the ecliptic, or if it always retained the fame po-
fition: But as this plane continually varies its pofition,
and its pole defcribes a fmall circle about the pole of
the ecliptic, from whence it is always about 5 degrees
diftant; it ftill remains that we examine what is the

pre-

preceffion caufed by the moon, with refpect to the ecliptic; which may be done by refolving the mo-tion of the earth's pole, parallel to the plane of the lunar orbit, into two motions, one whereof is parallel to the plane of the ecliptic, and the other perpen-dicular thereto, and which varies the inclination of the earth's axe to the ecliptic. This is what we fhall determine in the following *Problem*.

Problem VIII.

20. *Having given, the motion of the pole* P *of the earth parallel to the plane of the lunar orbit, to find its motion parallel to the plane of the ecliptic.*

* C P, fig. 3. n° 1. *& n*° 2. is the femi-axe of the earth. The plane of the figure is the plane of the folfticial colure; $C Z$ is the axe of the ecliptic, confequently perpendicular to $C \circledS$; $\circledS$ the folfticial point, and $P C \circledS$ the folfticial angle: $P Z$ is parallel to $C \circledS$, and perpendicular to $C Z$; and it is equal to the line $C H$ of *fig.* 2. The circumference l, l', l^2, l^3, l is the circumference defcribed by the pole, l, of the lunar orbit; L, L', L^2, L^3 is the fection of the cone $C l, l' l^2 l^3 l$ prolonged, by the plane that is tangen-tial at the pole of the earth, P.

Let l^o be any place of the pole of the moon's orbit. If upon the point P be erected $P R$, perpendicular to the plane $P C l^o$, this perpendicular will be the di-rection of the motion of the earth's pole, parallel to

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* The lines which are in the plane which touches the earth's pole l, are feparately reprefented in *fig.* 2. n° 3.

the

the plane of the lunar orbit, the plane $PC\,l^\circ$ being perpendicular to that plane, $PC\,l^\circ$ being the axe thereof.

Letting fall from the point R the perpendicular RM upon the line $\mathfrak{s}\,PZ$, the motion of the pole of the earth according to PR may be refolved into two; the one according to PM, the other parallel and equal to RM; and this latter exprefles the preceffion with regard to the ecliptic, and the former gives the alteration of the inclination of the earth's axe to the plane of the ecliptic.

The motion of the pole reprefented by MR, which gives the preceffion with regard to the ecliptic, is always on the fame fide during the entire revolution of the pole of the lunar orbit, and is the fame on either fide the point l at the like diftance from it; the motion according to PR being then the fame, and always on the fame fide, as has been explained in *Prob.* V.

* The motion according to PM is the fame on either fide of the point l at the fame diftance from it, but with a contrary fign, that is, in an oppofite direction. For it is manifeft, that if another point, as l°, be taken on the other fide, and at an equal diftance from the point l, the point M, where the perpendicular RM falls, will be on the other fide with regard to P, than the point M, fince the angle RPZ', which is here obtufe, will then be acute, and lefs than the right angle of an angle equal to $L^\circ\,PZ'$.

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* The point l° placed on the other fide of the point l, that is, between l and l^3, is not marked in the *fig.* nor the point M' where the perpendicular KM' falls; for the fake of not embarraffing the *fg.* with too many lines, it being eafy to conceive.

For

For this reason, the motion according to PM muſt produce an ofcillation of the earth's axis, whofe period is the time of an entire revolution of the pole of the lunar orbit about the pole of the ecliptic.

PROBLEM IX.

21. *To find the quantity of the preceſſion caufed by the moon with regard to the ecliptic; that is, the arc parallel to the ecliptic, run through by the earth's pole* P, *during the time of a femi-revolution* $1\ 1^r\ 1^s$ *of the pole* 1 *of the lunar orbit.*

Let there be drawn the lines NK, $l^o\ K$ perpendicular to the axe of the earth CP, and parallel to the lines NP, $l^o\ \mathcal{P}$, *fig.* 3. *n*o 1. & *n*o 2.

The triangles PRM, $\mathcal{P}NL^o$, KNl^o are right-angled and fimilar; for the three fides of the firſt are perpendicular to the three fides of the fecond, and the three fides of the fecond are parallel to the three fides of the third.

Still retaining the foregoing denominations, let $z\ l = x$; $ZN = z$; $\mathcal{P}Z = g$, as in *fig.* 2. the line CH; $CZ = f$; Cl or $Cl^o = C$.

We fhall have $Nl^o = \sqrt{xx - zz}$; $NP = g - z$; $NK = \dfrac{f}{a} \times \overline{g - z}$, or $\dfrac{f}{a} \times \overline{g + z}$, when the point l^o

lies on the other fide of l^r; $PR = \dfrac{g}{a} \times \overline{g - z}$, or $\dfrac{g}{a} \times$

$\overline{g + z}$ when the point l^o is on the other fide of l^r. The triangle $Cl^o K$ is here fimilar to the triangle whofe three fides are reprefented by the letters a, b, r, fince

since $\frac{b}{r}$ expresses here the sine of the angle of inclination of the earth's axe to the plane of the lunar orbit, and $\frac{a}{r}$ is its cosine; we shall have then $l° K =$

$$\frac{ac}{r}; \ C\,K = \frac{bc}{r} = C\mathcal{P} - \mathcal{P}K = a - \frac{g}{a} \times \overline{g-z} = $$

$$\frac{aa-gg+gz}{a} = \frac{ff+gz}{a}, \text{ or } \frac{ff-gz}{a}, \text{ when the}$$

point $l°$ is on the other side with respect to l^1; $\dfrac{RM}{PR}$

$$= \frac{NK}{l°K} = \frac{\frac{f}{a} \times \overline{g-z}}{\frac{ac}{r}}, \text{ or } \frac{\frac{f}{a} \times \overline{g+z}}{\frac{ac}{r}} \text{ for the other side}$$

of l^1; $\dfrac{\overline{PM}}{PR} = \dfrac{Nl°}{l°K} = \sqrt{\dfrac{\overline{xx-zz}}{\frac{ac}{r}}}$: Lastly, the differ-

ence of the arc $l^1 \, l° = d\,u = \dfrac{x\,dz}{\sqrt{\overline{xx-zz}}} = \dfrac{2\pi x}{nT}$

$\times d\,t$, calling $n\,T$ the time which the pole of the lunar orbit employs in performing its revolution in the circumference $2\,\pi x$, whence we get $d\,t = \dfrac{n\,T}{2\,\pi} \times$

$$\frac{dz}{\sqrt{\overline{xx-zz}}}.$$

The space run through in an instant $d\,t$ by the pole P of the earth parallel to the plane of the lunar orbit being

being by *Art.* 17. $= B\,b \times \dfrac{a\,a}{2\,r} \times d\,t$; and by *Prob.* VIII. the fpace run through parallel to the plane of the ecliptic, will be $B\,b \times \dfrac{aa}{2rr} \times d\,t \times \dfrac{\varkappa\,I}{P\,\varkappa} =$ (by the preceding denominations) $B\,b \times \dfrac{aa}{2rr} \times \dfrac{nT}{2\pi} \times \dfrac{d\,z}{\sqrt{\varkappa\varkappa - zz}}$

$$\times \dfrac{\dfrac{f}{a} \times \overline{g - z}}{\dfrac{a\,c}{r}} = B \times \dfrac{b}{r} \times \dfrac{nT}{4\pi} \times \dfrac{d\,z}{\sqrt{\varkappa\varkappa - zz}} \times \dfrac{f \times \overline{g - z}}{c};$$

and putting inftead of $\dfrac{b}{r}$, its value, which was found to be $\dfrac{ff + gz}{a\,c}$, we fhall have $B \times \dfrac{nT}{4\pi} \times \dfrac{f}{acc} \times \overline{ff + gz}$

$\times \overline{g - z} \times \dfrac{d\,z}{\sqrt{\varkappa\varkappa - zz}}$; and when the point l^0 is on the other fide with regard to l^1, we fhall have $B \times \dfrac{nT}{4\pi}$

$\times \dfrac{f}{acc} \times \overline{ff - gz} \times \overline{g + z} \times \dfrac{d\,z}{\sqrt{\varkappa\varkappa - zz}}$. Therefore always taking the preceffion for the two points together placed on either fide of l^1, and at an equal diftance from it, we fhall have, by joining the two values now found, after making reduction, $B \times \dfrac{nT}{4\pi} \times \dfrac{f}{aa} \times$

$\overline{2\,g\,ff - 2\,g\,zz} \times \dfrac{d\,z}{\sqrt{\varkappa\varkappa - zz}}$, or $B \times \dfrac{nT}{2\pi} \times \dfrac{fg}{aa} \times$

$\overline{ff - zz} \times \dfrac{d\,z}{\sqrt{\varkappa\varkappa - zz}}$. But

But the integral of $ff \times \dfrac{dz}{\sqrt{xx - zz}} = \dfrac{ff}{x} \times$

$\dfrac{dz}{\sqrt{xx - zz}}$ is $\dfrac{ff}{x} \times \dfrac{x\,\pi}{2} = \dfrac{ff\,\pi}{2}$, when we take

all the points l°, from l^1 to l; and the integral of

$\dfrac{zz\,dz}{\sqrt{xx - zz}}$, which is $-z\sqrt{xx - zz} + \int dz$

$\sqrt{xx - zz}$; when, after the integration, we make

$z = x$, becomes $\int dz \sqrt{xx - zz}$, or the area of

the quadrant $l^1\,z\,l$, which is $= \dfrac{x\,\pi}{2} \times \dfrac{x}{2} = \dfrac{x\,x\,\pi}{4}$.

Therefore the integral of $B \times \dfrac{nT}{2\pi} \times \dfrac{fg}{acc} \times \sqrt{ff - zz}$

$\times \dfrac{dz}{\sqrt{xx - zz}}$ is $= B \times \dfrac{nT}{2\pi} \times \dfrac{fg}{acc} \times \left(\dfrac{ff\,\pi}{2} - \dfrac{x\,x\,\pi}{4} \right)$

$= B \times \dfrac{nT}{2} \times \dfrac{fg}{acc} \times \left(\dfrac{ff}{2} - \dfrac{xx}{4} \right) = B \times \dfrac{nT}{2} \times \dfrac{gf^3}{2acc} \times$

$\left(1 - \dfrac{xx}{2ff} \right)$.

This is the fum of all the inftantaneous preceffions for all places of the lunar orbit from l to l^2, becaufe we have always taken two points together placed on either fide of l^1.

And this is the quantity of the preceffion with regard to the ecliptic, during the time of a femi-revolution of the lunar orbit. Which was to be found.

PROBLEM X.

22. *To find the quantity of the nutation, or of the variation of the inclination of the earth's axe to the ecliptic, caused by the moon during the semi-revolution* $1\,l'\,l^2$ *of the pole of the lunar orbit.*

The space run through in an inftant $d\,t$ by the pole of the earth, parallel to the plane of the lunar orbit, being, by *Art.* 17. $= B\,b \times \dfrac{aa}{2rr} \times d\,t$, the inftantaneous nutation for any one place l° of the lunar orbit will be, by *Prob.* VIII. $= B\,b \times \dfrac{aa}{2\,rr} \times d\,t \times \dfrac{PM}{PR}$

$=$ (by the denomination of the preceding *Problem*)

$$B\,b \times \frac{aa}{2rr} \times \frac{nT}{2\pi} \times \frac{d\,z}{\sqrt{xx - zz}} \times \frac{\sqrt{xx - zz}}{\dfrac{a\,c}{r}} = B \times \frac{b}{r}$$

$$\times \frac{aa}{2} \times \frac{nT}{2\pi} \times \frac{dz}{\sqrt{xx - zz}}.$$ Putting inftead of $\dfrac{b}{r}$ its

value, which was found to be $\dfrac{ff + g\,z}{a\,c}$, we shall have

$$B \times \frac{n\,T}{2\pi \times 2\,cc} \times \overline{ff + g\,z} \times d\,z\,;$$ and when the point

$l^\bullet$ is on the other fide with regard to l', we shall have

$$B \times \frac{n\,T}{2\pi \times 2\,c\,c} \times \overline{ff - g\,z} \times d\,z.$$

Taking therefore the nutation for the two points together, l°, placed on either fide of l', we have

$$B \times$$

$B \times \dfrac{nT}{2\pi \times 2cc} \times 2ffdz$, whose integral, when z is

$= x$, is $B \times \dfrac{nT}{2\pi \times 2cc} \times 2ff\,x = B \times \dfrac{nT}{2\pi} \times \dfrac{ff}{cc} \times x.$

This is the quantity of the nutation caused by the moon during the semi-revolution $l\,l^1\,l^2$ of the pole of the lunar orbit. Which was to be found.

COROLLARY.

23. By *Prob.* IX. the quantity of the precession caused by the moon, with regard to the ecliptic, is, during the space of a semi-revolution $l\,l^1\,l^2$ of the pole of the lunar orbit, $B \times \dfrac{nT}{2} \times \dfrac{gf^3}{2acc} \times \left(1 - \tfrac{1}{2} \times \dfrac{xx}{ff}\right).$

By the preceding *Problem*, the quantity of the nutation during the same time of a semi-revolution of the pole of the lunar orbit $l\,l^1\,l^2$, caused also by the moon, is $B \times \dfrac{nT}{2\pi} \times \dfrac{ff}{cc} \times x.$

Therefore the precession, during the time of a semi-revolution of the pole of the lunar orbit, is to the nutation, during the same time,

As $\dfrac{gf}{2a} \times \left(1 - \tfrac{1}{2}\dfrac{xx}{ff}\right)$, is to $\dfrac{x}{\pi}$; or as $gf\pi \times$

$\left(1 - \tfrac{1}{2}\dfrac{xx}{ff}\right)$ is to $2\,ax$.

But the precession is performed in a parallel to the ecliptic, whereof g is the radius; and the nutation is performed in the plane of the solsticial colure, and in a circle whereof a is the radius: Di-

viding,

viding, therefore, each circular arc by its radius, we shall have the quantity of the angle : And

The angle of the preceffion, during the time of a semi-revolution of l l' l^2 of the pole of the lunar orbit, will be to the angle of the nutation, in the same time,

As $f \pi \times \left(1 - \frac{1}{2} \frac{xx}{ff}\right)$ is to $2x$, or as $\pi \left(1 - \frac{1}{2} \frac{xx}{ff}\right)$ is to $\frac{2x}{f}$.

SCHOLIUM.

24. Whatever has been said as to the moon and the fun, may be equally applicable to all the other planets; but as the effect of the action of the reft of the planets upon the earth is not fenfible, we shall not take it into confideration.

SECT. III.

Of the motion of the pole of a ring, caufed by the action of the fun: or of the motion of the pole of the moon's orbit.

25. We have feen, in *Art.* 9. that if, inftead of the terreftrial fpheroid, we were to fuppofe a fimple ring placed at the circumference of the equator, the motion of the pole of fuch ring would be found by the fame *formula* as that for the motion of the pole of the earth; fuppofing only, in that *formula*, $\frac{a}{1-2a} = 1$.

By

By this means we may apply all the *formulæ* which have been found in *Sect.* I. to the motion of the nodes of this ring, and to the alteration of the inclination of its axe to the plane of the ecliptic.

But by the *Remark* on *Art.* 6. the motion of the plane of this ring is the fame, whether the ring be entire, or there be only a fingle point which circulates in the ring's circumference : Whence it follows, that

Thefe fame *formulæ* do likewife give the motion of the pole of the orbit of a moon or fatellite; the motion of the node of the orbit of fuch moon in the ecliptic ; and the variation of the inclination of its axe of the orbit to the plane of the ecliptic : Obferving to put the time of the revolution of this moon about the fun, inftead of the time of the earth's revolution about the fun, and the motion of the moon in its orbit inftead of the motion of a point of the equator.

Remark.

It may be obferved, that although the motion of the pole of a ring be the fame as that of a moon, during the time of the revolution of the ring, or of the moon, which is the fame ; yet there are fome particular motions which take place when there is only one moon revolving in the circumference of the ring, and which ceafe to exift when it is an entire ring that revolves.

For example; in *Art.* 2. the force, according to $L \gamma$, *fig.* 1. n° 2. of the point L, is deftroyed by an equal and directly oppofite force of another point λ placed on the other fide, and at the fame diftance from the point E, as the point L, when the ring is entire;

but

but when there is but one point, as L, or, which is the fame, one moon, which revolves in the circumference of the ring; this motion fubfifts, and difturbs the motion of the moon in the plane of its orbit.

In like manner the force of the point L parallel to $\gamma\, C$, or according to $c'\, L$, is deftroyed, in a ring, by the force of another equal point λ' placed on the other fide with regard to ♃, and at the fame diftance as the point L. But this force fubfifts in the cafe where there is but one fingle point L, or one moon, and this motion diforders, befides, the motion of the moon in the plane of its orbit.

Laftly, If the ring, which circulates about the centre C, be elliptical, the motion of every point of the ring in the orbit is proportional to its diftance from the centre; becaufe, on account of the continuity of the ring, all the points complete their revolution in the fame time, and have equal angular motions; but when it is but one fingle moon that revolves, in an elliptic orbit, it defcribes areas proportional to the times, and has an angular motion fo much the greater, as it approaches nearer the centre; which is the contrary of the motion of the points of the elliptic ring, the more diftant whereof from the centre, have the moft motion. And it arifes from hence, that the motion of the pole of the lunar orbit is much the fame as that of the ring, taking it during the entire revolution; but that there is a fmall variation, or inequality, in the inftantaneous motion of the pole of a moon; whereas this inftantaneous motion is conftant through every inftant of the revolution of the ring.

I fhall

I ſhall ſpend no more time on the ring, with re-
ſpect to thoſe motions which have nothing to do with
the ſubject in queſtion ; but they ſhall be thoroughly
diſcuſſed, and applied, in a ſecond memoir, which I
ſhall publiſh, on the particular motions of the planets.

S E C T.   IV.

Containing the application of the formulæ *of
the preceding Sections.*

26. The motion of the ſun, or, to ſpeak more ex-
actly, of the earth, in the ecliptic, is $\dfrac{2\pi s}{T}$, which is
the ſpace divided by the time.

27. The earth's central force will therefore be
$\left(\dfrac{2\pi s}{T}\right)^{2} \times \dfrac{1}{2\,s}$, the central force being equal to the
ſquare of the motion divided by the diameter of the
circumference in which the body moves; and the
earth's central force being the force $\dfrac{S}{s\,s}$ wherewith it

is attracted towards the ſun ; we ſhall have $\dfrac{S}{s\,s} =$

$\left(\dfrac{2\pi s}{T}\right)^{2} \times \dfrac{1}{2\,s} = \dfrac{2\pi\pi s}{T\,T}$; whence we get $\dfrac{S}{s^{3}} = \dfrac{2\pi\pi}{T\,T}$.

28. The motion of a point of the equator whoſe
radius is *a*, will be, calling the time of its revolution
$t, = \dfrac{2A\pi}{t}$, which is the ſpace divided by the time.

We have therefore $m = \dfrac{2A\pi}{t}$.

29. By *Art.* 8. $A = \frac{3S}{s^3 m} \times \frac{aa}{1-2a} =$ (by *Articles*

27. & 28.) $3 \times \frac{2\pi\pi}{TT \times 2a\pi} \times \frac{aa}{1-2a} = \frac{3t}{TT} \times \pi \times \frac{a}{1-2a}$

$=$ (by *Art.* 31. following) $\frac{3t}{TT} \times \pi \times \frac{1}{176}$.

30. Putting q for the ratio of the lunar force $\frac{L}{\lambda^3}$, to the folar force $\frac{S}{s^3}$, we fhall have $\frac{L}{\lambda^3} = \frac{S}{s^3} \times q$; and B, which (by *Art.* 16.) is $= \frac{3L}{\lambda^3 m} \times \frac{aa}{1-2a}$ will alfo be

$= \frac{3S}{s^3 m} \times q \times \frac{aa}{1-2a} =$ (by *Art.* 29.) $\frac{3qt}{TT} \times \pi \times \frac{1}{176}$.

31. Suppofing $a = \frac{1}{178}$, which is conformable to the obfervations of the gentlemen of the *Parifian* academy, we fhall have $\frac{a}{1-2a} = \frac{1}{178 \times 1 - \frac{2}{178}} = \frac{1}{176}$.

PROBLEM XI.

32. *To find the preceffion caufed by the fun during the time between the equinox and the folftice; that is to fay, in a quarter of a year, or three months.*

By *Art.* 12. the preceffion, during this time, is $Ab \times \frac{T}{2} \times \frac{g}{4r}$. Putting, inftead of A, the value

found

found in *Art.* 29. we fhall have $\dfrac{3\,t}{T\,T}\times\pi\times\dfrac{1}{176}\times\dfrac{b\,T}{2}$

$\times\dfrac{g}{4\,r}=\dfrac{3\,t}{T}\times\dfrac{1}{176}\times\dfrac{b}{r}\times\dfrac{\pi\,g}{8}$.

But $\dfrac{b}{r}$ is the ratio of the fine of the inclination of the axe of the earth to the ecliptic, to the radius, or of the fine of 66 degrees 32 minutes to the radius, $=\dfrac{9172919}{10000600}$; and $\pi\,g$ is the femi-circumference of the parallel to the ecliptic wherein the pole of the earth moves, and, confequently, equal to 180 deg.

t, which is the time of the revolution of a point of the equator, is equal to one day; and T equal to $365\frac{1}{4}$ days; we fhall have, then, by fubftituting all the values in the preceding *formulæ*, $\dfrac{3}{365\frac{1}{4}}\times\dfrac{9172919}{10000000}\times$

$\dfrac{1}{176}\times\dfrac{180}{8}$ degrees. This is the preceffion produced by the fun during the fpace of 3 months.

33. Multiplying by 4, we fhall have the annual preceffion caufed by the fun $=\dfrac{3}{365\frac{1}{4}}\times\dfrac{9172919}{10000000}\times$

$\dfrac{1}{176}\times\dfrac{180}{2}$ degrees, $=\dfrac{3}{365\frac{1}{4}}\times\dfrac{9172919}{10000000}\times\dfrac{45}{88}$ degrees, $=$ (after due reduction) $13''.\ 52'''.\ 11''''$.

PROBLEM XII.

34. *To find the precession caused by the moon, during the time of a semi-revolution of the pole of the lunar orbit about the pole of the ecliptic; that is, in about* $9\frac{1}{3}$ *years, or* $9\frac{1}{3} \times$ T, *as* T $=$ 1 *year.*

By *Art.* 21. we have found this precession $= B \times \frac{nT}{2} \times \frac{g f^3}{2acc} \times \left(1 - \frac{1}{2} \times \frac{xx}{ff}\right)$.

By *Art.* 30. $B = \frac{3qt}{TT} \times \pi \times \frac{1}{176}$: Substituting this value in the preceding *formula*, it becomes $\frac{3qt}{T} \times \frac{n}{2} \times \frac{f}{2a} \times \frac{ff}{cc} \times \frac{1}{176} \pi g \times \left(1 - \frac{1}{2}\frac{xx}{ff}\right)$.

But $\frac{f}{a}$ is the ratio of the fine of 66 deg. 32 min. to the radius, $= \frac{9172919}{10000000}$.

$\frac{f}{c}$ is the ratio of the fine of 85 deg. to the radius, the angle $l\,c\,z$, whereby the pole of the lunar orbit is distant from the pole of the ecliptic, being, at all times, about 5 degrees. Therefore $\frac{f}{c} = \frac{9961947}{10000000}$.

$\frac{x}{f}$ is the ratio of the fine of 5 deg. to the fine of 85 degrees, $= \frac{871557}{9961947}$.

3   Therefore

Therefore fubftituting thefe values, and thofe of
t, T, and πg found in the preceding *Problem*, we
fhall find that the preceffion, during a femi-revolution
of the lunar pole, which is $\dfrac{nT}{2}$, by *Art.* 21. becomes

$$= q \times \frac{n}{2} \times \frac{3}{365\frac{1}{4}} \times \frac{9172919}{10000000 \times 2} \times \left(\frac{9961947}{10000000} \right)^2 \times \frac{1}{176}$$

$$\times 180 \text{ deg.} \times \left(1 - \tfrac{1}{2} \times \left(\frac{871557}{9961947} \right)^2 \right).$$

This is the preceffion caufed by the moon during
the time of a femi-revolution of the pole of the lunar
orbit, $\dfrac{nT}{2} = 9\frac{1}{3} \times T$.

35. Therefore the mean preceffion caufed by the
moon during the time T, which is the mean annual
preceffion caufed by the moon, will be, by dividing
the preceding *formula* by $\dfrac{n}{2}$, $= q \times \dfrac{3}{365\frac{1}{4}} \times \dfrac{9172919}{10000000 \times 2}$

$$\times \left(\frac{9961947}{10000200} \right)^2 \times \frac{180 \text{ deg.}}{176} \times \left(1 - \tfrac{1}{2} \times \left(\frac{871557}{9961947} \right)^2 \right)$$

where it only remains to determine the value of q,
which, by *Art.* 30. is the ratio of the lunar force to
the folar force; and making this ratio $= \dfrac{5}{2}$, conform-
able to the obfervations of the tides related by Mr.
Daniel Bernoulli, we fhall have for the mean annual
preceffion caufed by the moon, $\dfrac{5}{2} \times \dfrac{3}{365\frac{1}{4}} \times \dfrac{9172919}{10000000 \times 2}$

$$\times \left(\frac{9961947}{10000000} \right)^2 \times \frac{180 \text{ deg.}}{176} \times \left(1 - \tfrac{1}{2} \times \left(\frac{871557}{9961947} \right)^2 \right)$$

$=$ (after due reduction) $34''. 16'''. 51''''$.

Pro-

PROBLEM XIII.

36. To find the nutation of the earth's axe caused by the moon, during the time of a semi-revolution of the pole of the moon's orbit, that is, in 9½ years.

By *Art.* 23. $\pi \times \left(1 - \frac{1}{2} \times \frac{x\,x}{f\,f}\right)$ is to $\frac{2\,x}{f}$, as the angle of the preceſſion, found in the preceding *Problem* for the time $\frac{n\,T}{2}$ of the ſemi-revolution of the pole of the lunar orbit, is to the nutation during the ſame time. By it, therefore, may be found the nutation, having the quantity of the preceſſion of the preceding *Prob.* or, reciprocally, knowing the nutation during the time of a ſemi-revolution of the pole of the lunar orbit, the preceſſion will be known to a geometrical certainty, for the ſame time. The ratio, in numbers, is $\frac{1095798}{61236}$.

37. But we may alſo know the nutation during the time of a ſemi-revolution of the pole of the lunar orbit, by the *formula* of *Art.* 22. which is $B \times \frac{n\,T}{2} \times \frac{f\,f}{c\,c} \times x$; which, without alteration, may be multiplied by $\frac{a\pi}{a\pi}$, or by $\frac{a\,\pi}{a \times \frac{22}{7}}$; ſince π, which is the ratio of the circumference to the diameter, is $\frac{22}{7}$, and we ſhall have

have the *formula* $B \times \dfrac{nT}{2} \times \dfrac{ff}{cc} \times x \times \dfrac{a\,\pi}{a \times \dfrac{22}{7}} = B \times \dfrac{nT}{2}$

$\times \dfrac{f}{a} \times \dfrac{f}{c} \times \dfrac{x}{c} \times \dfrac{7}{22} \times a\,\pi$; which, by fubftituting the values of the preceding *Problem*, and 180 degrees inftead of $a\,\pi$, which is the femi-circumference of the circle wherein the nutation is performed, be-comes $\dfrac{3qt}{T} \times \dfrac{n}{2} \times \dfrac{9172919}{10000000} \times \dfrac{9961947}{10000000} \times \dfrac{871557}{10000000} \times$

$\dfrac{7}{22} \times \dfrac{180}{176}$deg. $= \dfrac{3}{365\frac{1}{4}} \times \dfrac{5}{2} \times 9\frac{1}{2} \times \dfrac{9172919}{10000000} \times \dfrac{9961947}{10000000}$

$\times \dfrac{871557}{10000000} \times \dfrac{7}{22} \times \dfrac{180}{176}$ deg. equal, after reduction, to $17''. 51'''. 14''''$.

PROBLEM XIV.

38. *To find the difference of the inclination of the earth's axe, between the point of the equinox and that of the folftice, caufed by the action of the fun.*

By *Art.* 14. the *formula* is $\frac{1}{2} A \times \dfrac{T}{2\pi r} \times a\,a$.

Putting, inftead of A, its value of *Art.* 29. inftead of $\dfrac{a}{r}$, which is the ratio of the fine of the obliquity of the ecliptic to the radius $\dfrac{3982155}{10000000}$, we fhall have

$\frac{1}{2} \times \dfrac{3}{365\frac{1}{4}} \times \dfrac{1}{2 \times 176} \times \dfrac{3982155}{10000000} \times a$; and putting,

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in-

inſtead of a, $\dfrac{a\pi}{\pi} = \dfrac{a\pi}{\frac{22}{7}} = \dfrac{7}{22}\, a\,\pi = \dfrac{7}{22} \times 180$ deg.

(ſince $a\,\pi$ is the ſemi-circumference of the circle wherein this inclination is performed) we ſhall have $\dfrac{3}{365\frac{1}{4}} \times \dfrac{1}{4 \times 176} \times \dfrac{3982155}{10000000} \times \dfrac{7}{22} \times 180$ deg. $=$ (after reduction) $57'''.\ 25''''$.

PROBLEM XV.

39. *To find the difference of the inclination of the earth's axe to the plane of the lunar orbit, when the moon arrives at its point of ſtation, 90 degrees diſtant from the node of the equator with the lunar orbit, this difference of inclination being cauſed by the moon.*

By *Art.* 18. the variation of inclination will be $\frac{1}{2} B \times \dfrac{M}{2\pi r} \times a\,a.$

Putting, inſtead of B, its value of *Art.* 30.; in-inſtead of M, which expreſſes the time of the moon's revolution about the earth, $27\frac{1}{3}$ days; in-ſtead of $\dfrac{a}{r}$, the mean value $\dfrac{3982155}{10000000}$; I ſay *mean value*, becauſe the earth's axe is not always equally inclined to the plane of the lunar orbit, as has been ſhewn in *Art.* 19. *&c.* we ſhall have $\frac{1}{2} \times \dfrac{3 \times \frac{5}{2}}{(365\frac{1}{4})^2} \times$

$\dfrac{27\frac{1}{3}}{2 \times 176} \times \dfrac{3982155}{10000000} \times a =$ (putting, as in the pre-

ceding

ceding *Problem*, inftead of *a*, $\frac{7}{22}$ × 180 degrees) $\frac{1}{2}$ ×

$$\frac{3\frac{5}{7}}{(365\frac{1}{4})^2} \times \frac{27\frac{1}{3}}{2 \times 176} \times \frac{3982155}{10000000} \times \frac{7}{22} \times 180 \text{ degrees,} =$$

(after reduction) 10‴. 19⁗.

PROBLEM XVI.

40. *To find the motion of the node of* Jupiter's *fourth fatellite, caufed by the fun, during the time of* $\frac{1}{4}$ *of a revolution of that fatellite about the fun.*

We need only make ufe of the *formula* of *Problem* XI. wherein, conformably to *Art.* 25. we muft make $\frac{a}{1-2a} = 1$, $t = 16\frac{3}{4}$ days, being the time of the revolution of the fatellite about *Jupiter*; $T = 365\frac{1}{4} \times 12$ days, which is the time of the fatellite's revolution about the fun, or about 12 years; $\frac{b}{r}, = 1$,

becaufe the orbit of the fatellite is but little inclined to the ecliptic, and, confequently, the axe thereof nearly at right angles thereto.

We fhall have then $\frac{3t}{T} \times \frac{\pi g}{8}$, or $\frac{3 \times 16\frac{3}{4}}{365\frac{1}{4} \times 12} \times \frac{180 \text{ deg.}}{8}$.

This is the motion of the node of this fatellite in 3 years, or during $\frac{1}{4}$ of its revolution about the fun.

Remark.

41. By this may be found the motion of the nodes of the reft of *Jupiter's* fatellites. For it appears, from the *formula*, that the motion of the nodes of

<div align="right">different</div>

different fatellites of the fame planet, is as the times of their revolutions about the planet.

Thus the motion of the nodes of the 3d fatellite, in the fpace of 3 years, is $\dfrac{3 \times 7\frac{1}{6}}{365\frac{1}{4} \times 12} \times \dfrac{180 \text{ deg.}}{8}$.

The motion of the nodes of the 2d is $\dfrac{3 \times 3\frac{11}{24}}{365\frac{1}{4} \times 12}$ $\times \dfrac{180 \text{ deg.}}{8}$.

The motion of the nodes of the 1ft is $\dfrac{3 \times 1\frac{3}{4}}{365\frac{1}{4} \times 12}$ $\times \dfrac{180 \text{ deg.}}{8}$.

PROBLEM XVII.

42. *To find the motion of the nodes of the 5th fatellite of* Saturn, *caufed by the fun, during the time of ¼ of the revolution of this fatellite about the fun.*

Conformably to the preceding *Problem* we fhall have, by putting inftead of t, $79\frac{2}{3}$ days; inftead of ♄ $365\frac{1}{4} \times 30$ days, which is nearly the time of the revolution of this fatellite about the fun, and we fhall have $\dfrac{3 \times 79\frac{2}{3}}{365\frac{1}{4} \times 30} \times \dfrac{180 \text{ deg.}}{8}$.

43. In the fame manner may be found the motion of the nodes of all the reft of *Saturn's* fatellites.

Problem XVIII.

44. *To find the motion of the moon's node during the time that the sun is passing from the node of the moon's orbit with the ecliptic, to its point of station; that is, in three months, wanting a few days.*

Here we are to make use of the *formula* of *Art.* 13. because the motion of the moon's nodes is very sensible in respect of the motion of the sun.

We shall have $Ab \times \dfrac{T}{2r} \times \dfrac{g}{4} \times (1 - Ab \times \dfrac{T}{2\pi r} \times \frac{1}{4})$; and employing the values of *Prob.* XI. except that here $t = 27\frac{1}{3}$ days, which is the time of the moon's revolution about the earth, and that $\dfrac{b}{r} = $ to the sine of 85 deg. divided by the radius; the axe of the lunar orbit being inclined in an angle of about 85 degrees to the ecliptic. Therefore $\dfrac{b}{r} = \dfrac{9961947}{10000000}$. Lastly, by, *Art.* 25. we must take $\dfrac{a}{1 - 2a} = 1$, and the preceding *formula* will be changed into this, $\dfrac{3 \times 27\frac{1}{3}}{365\frac{1}{4}} \times \dfrac{9661947}{10000000}$ $\times \dfrac{180\,\text{deg.}}{8} \times (1 - \dfrac{3 \times 27\frac{1}{3}}{365\frac{1}{4}} \times \dfrac{9961947}{10000000} \times \dfrac{3}{2 \times 8}) = $ (after reduction) $4^\circ.\ 36'.\ 33''$.

This is the motion of the moon's node during the time that the sun is passing from the node of the moon's orbit with the ecliptic to its point of station; that is, during the time the sun is moving 90 — 4 degrees

grees, 36 minutes, 33 seconds, or $85°.23'.27''$. And if the motion of the node be required for one year, 'tis only multiplying the motion, just found, by $\dfrac{360°}{85°.23'.27''.}$, and it will be found $19°.25'.39''$. being a very small matter more than the truth, for reasons which will be explained elsewhere, and because we have taken $27\frac{1}{3}$ days for the time of the moon's revolution about the earth, which is really a little less than that quantity.

PROBLEM XIX.

45. *To find the greatest difference of the inclination of the lunar orbit to the ecliptic; which happens when the sun arrives at his point of station with regard to the moon.*

Here the *formula* of *Art.* 15. must be made use of, which gives $\frac{1}{2} A a a \times \dfrac{T}{2\pi r} \times \left(1 - \frac{1}{2} A b \times \dfrac{T}{2\pi r}\right)$, or,

which is the same, $\frac{1}{2} A \times \dfrac{aT}{2\pi r} \times \dfrac{a\pi}{\frac{22}{7}} \times \left(1 - \frac{1}{2} A \times \dfrac{bT}{2\pi r}\right)$,

substituting the values as in the preceding *Problem*, and instead of $\dfrac{a}{r}$, putting $\dfrac{871557}{10000000}$, which is the ratio of the fine of $5°$ to radius, we shall have $\dfrac{3 \times 27\frac{1}{3}}{365\frac{1}{4}} \times$

$\dfrac{1}{2 \times 2} \times \dfrac{871557}{10000000} \times \dfrac{7}{22} \times 180 \text{ deg.} \times \left(1 - \frac{1}{2} \times \dfrac{3 \times 27\frac{1}{3}}{365\frac{1}{4}}\right.$

$\left. \times \dfrac{9961947}{10000000}\right) = 15'.51''.$

Remark.

Remark.

46. It fhould be obferved, that, to fhorten the computations, I have contented myfelf with taking the times of the revolutions pretty near the truth; but if the utmoft exactnefs be required, the accurate times of the revolutions muft be employed.

47. This might be a proper place to add the method of determining the perturbation of the orbit of any planet, as derived from another planet; but fince this depends upon no other than the very fame principles that have been made ufe of in this memoir, and as their application will be fhewn, in its full extent, in the memoir which I am going to print, and intend myfelf the honour of fending to the *Royal Society*, I fhall defift, that I may not run this paper to a greater length.


LIX. *A Letter to the Right Honourable* George *Earl of* Macclesfield, *P. R. S. concerning the ages of* Homer *and* Hefiod. *By* George Coftard, *M. A.*
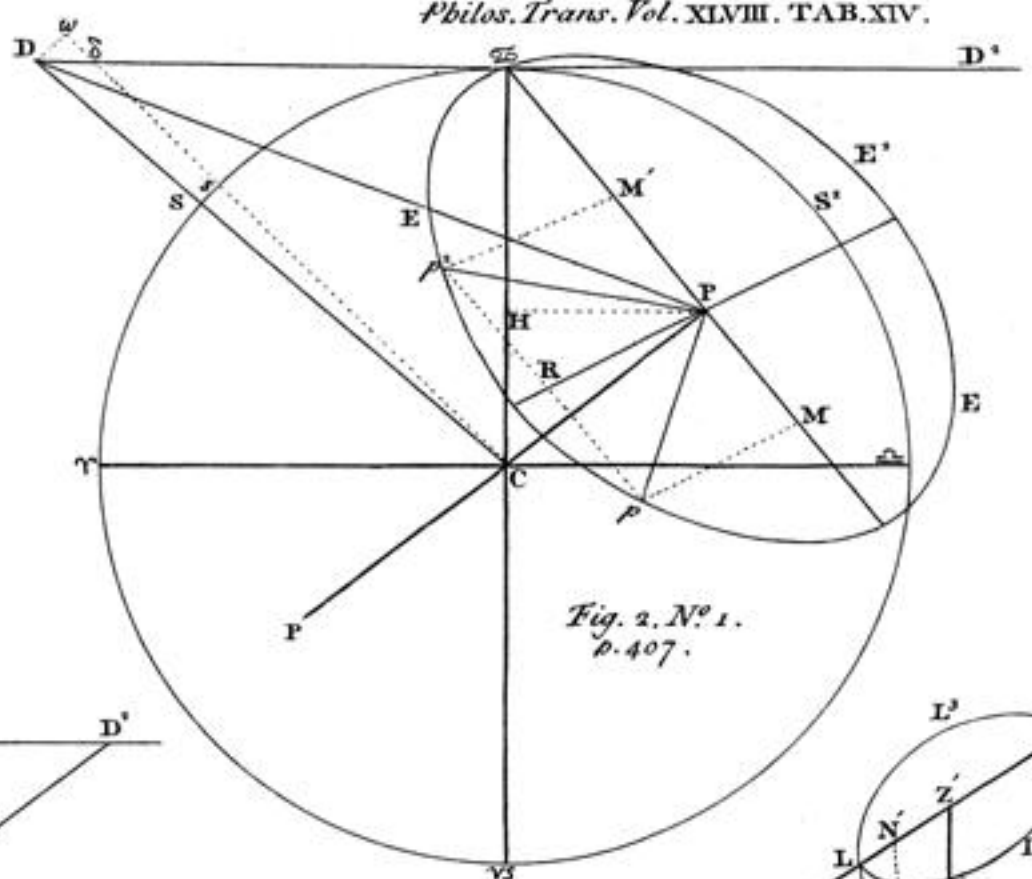
My Lord,

IT feems to be an opinion pretty generally received, that Homer and Hefiod lived much about the fame time. If this be true, and they did fo, whatever arguments prove the age of one, will equally ferve for fixing that of the other. What that age was, is indeed not at all agreed on among writers; the only thing in which they con-
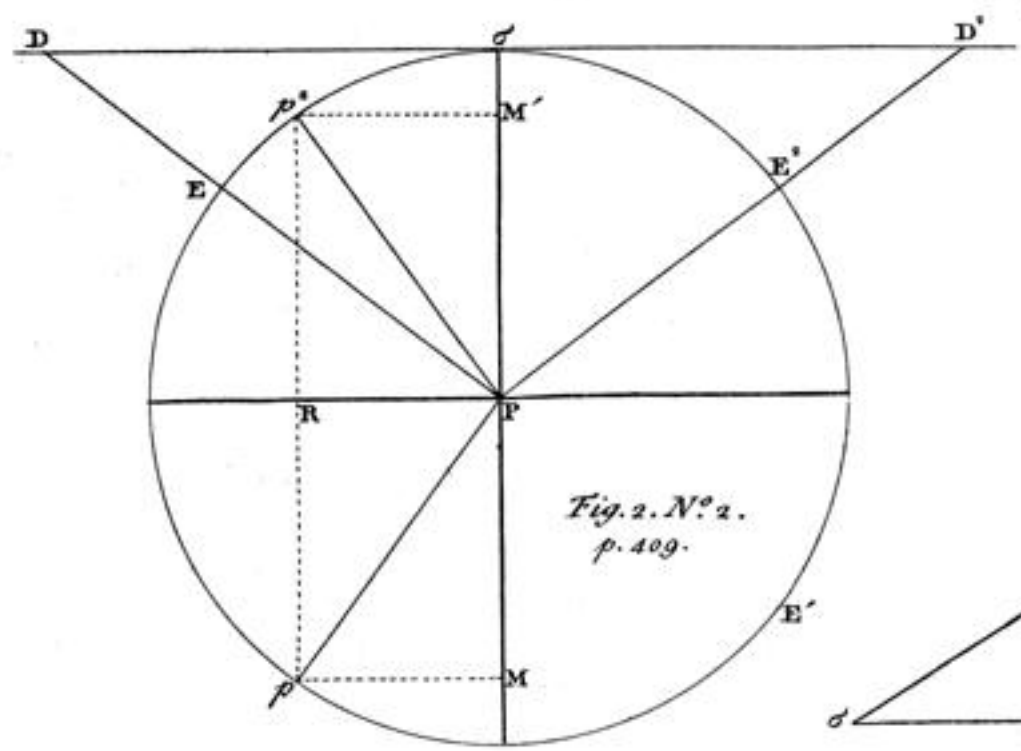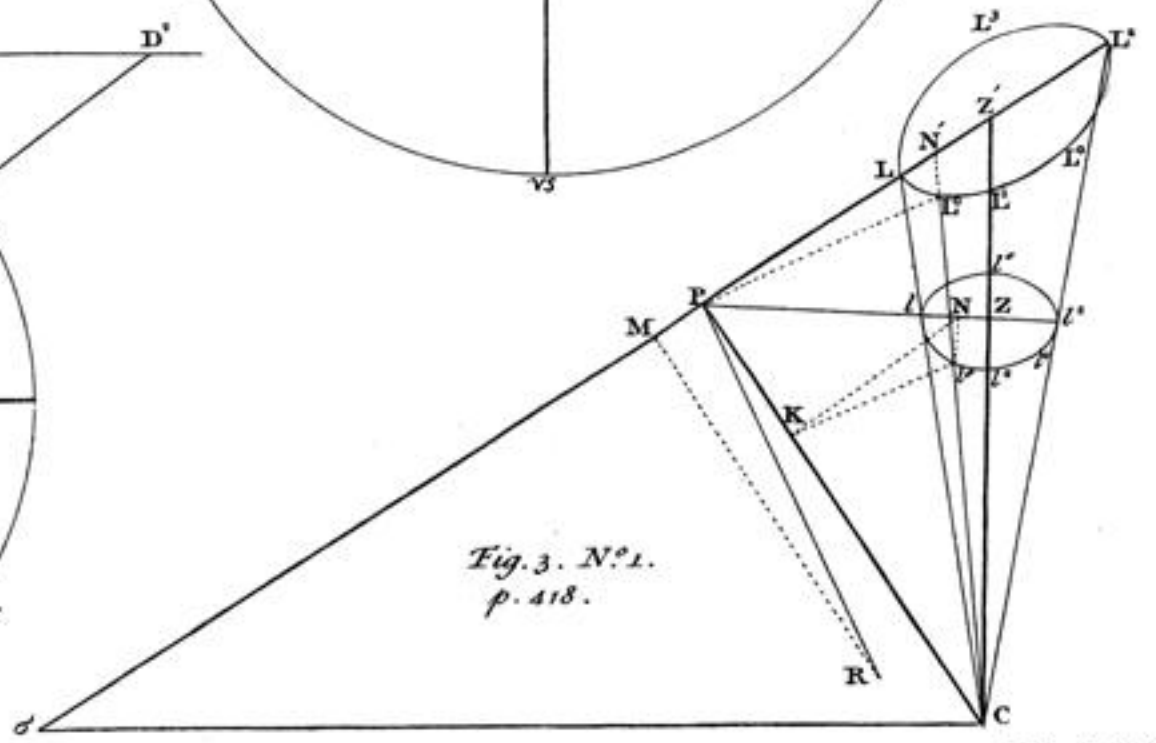
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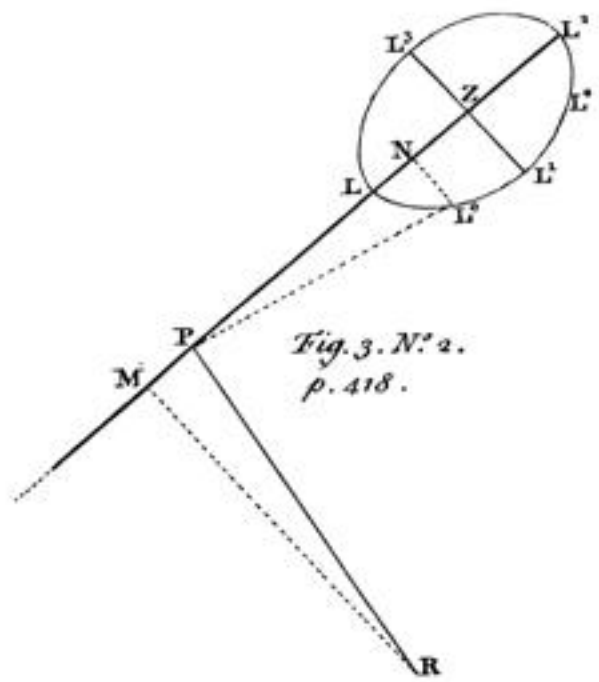
Fig. 3. N.º 2.
p. 418.

Fig. 2. N.º 1.
p. 407.

Fig. 2. N.º 2.
p. 409.

Fig. 3. N.º 1.
p. 418.

J. Mynde Sculp.